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THE LINEAR FUNCTIONS OF A COMPLEX VARIABLE.*

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I. THE INTEGRAL LINEAR FUNCTION.

1. The simplest possible functions of a complex variable, z, are obviously those which involve this variable rationally and only to the first degree. These are the linear functions of the general form, $(\alpha z + \beta) / (\gamma z + \delta)$, where α, β, γ, and δ are any complex constants whatever. These functions in general consist of a numerator and a denominator, each of which involves z, and are accordingly fractional functions of z. If, however, γ disappear from the denominator, the corresponding functions reduce to the form $(\alpha z + \beta) / \delta$, which, since α, β, and δ are any complex numbers, may be written simply $\alpha' z + \beta'$. These last expressions are integral linear functions of z. Any fractional linear function is the ratio of two integral linear functions.

We denote the function which may be at any instant under consideration by the single letter w, and accordingly write in the present case $w = (\alpha z + \beta) / (\gamma z + \delta)$. z being a complex variable, w is a complex variable also. To any system of values of z corresponds a system of values of w. If, now, any number of values of z be represented by a system of points in the complex plane, the corresponding values of w will be represented by a second system of points in the plane. The geometric relations between these two systems of points evidently exactly replace the analytical relations between the values of the variables z and w, as defined by the given equation. But the geometric relations have the great advantage of a concrete and clearly presentable form. Accordingly, we shall turn this method of geometric interpre-

* In the following series of articles I have attempted to give a clear presentation of the theory of the linear functions of a complex variable from a geometrical stand-point. On the part of the reader I have assumed only a knowledge of the ordinary geometrical representation and methods of combination of the complex number. For the sake of brevity, the plane in which the complex number is represented is called simply the complex plane.

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tation of functional relations to full account, and shall devote the present article to a systematic study of the linear functions from a geometrical point of view.

2. We begin with an extremely simple case, that of the integral linear function, $w = z + \beta$.

Given any point, z, in the complex plane, the corresponding point, w, is found, as we know, by adding to the vector z the vector β. We may regard the operation of passing from the point z to the point w as taking place in this way: that the point z moves continuously from its initial to its final position along the vector joining the two. Since all points in the plane are to undergo this same process, all the points w may be regarded as obtained from the corresponding point z by a *translation* of the entire plane through the vector distance β.

We shall accordingly regard the equation $w = z + \beta$ as defining this operation of translation. This mode of interpretation of a functional relation as defining a geometrical operation to be performed in the complex plane, by which we pass from the values of the variable to the corresponding values of the function, is the guiding principle of the entire present investigation.

3. A second simple case of the integral linear transformation is that of the function $w = az$, where a is any complex number, $a_1 + a_2 i$ or $\rho (\cos \varphi + i \sin \varphi)$.

Suppose, at first, that a is real; that is, that $\varphi = 0$. The corresponding operation in the complex plane then consists in multiplying the length of the vector of each point z by ρ, the vector remaining otherwise unchanged. The effect is evidently an *expansion* of the plane outward from the origin equally in all directions, or a contraction of the plane inward toward the origin equally in all directions, according as ρ is greater or less than unity. For simplicity we include both expansion and contraction under the one name expansion. The quantity ρ is called the *ratio* of the expansion.

For a more detailed consideration of the operation of expansion, suppose a system of rays drawn through the origin and a system of concentric circles described about the origin as a centre. Then the effect of the transformation is this: that every ray remains unchanged as a whole, while each circle is converted into a second circle of the system, with a radius equal to ρ times that of the original circle. Every point in the plane moves along the ray on which it lies. The rays are accordingly called the *lines of motion* of the transformation. The concentric circles cut the rays everywhere at right angles, and are therefore called the *orthogonal system* of curves for the transformation.

If we construct the system of concentric circles in such a way that the

ratio of the radii of each successive pair is ρ, the effect of the transformation will be to convert each of the circles into the next following one. Each of the curvilinear rectangles contained between two successive circles and two rays will be connected by the transformation into the adjacent rectangle between the same two rays farther outward or inward, according as $\rho \gtrless 1$. The dimensions of the rectangle being each multiplied by ρ, its area is multiplied by ρ^2

Again, if we consider any portion of the plane whatever, this will be carried away from the origin, retaining its form, until the distance of each point within it is ρ times its former distance, all the dimensions of the figure being consequently multiplied by ρ as before.

These are all thoroughly elementary matters, but they are of fundamental importance, and we therefore treat them at the outset in great detail. Evidently the construction of diagrams analogous to those just considered will furnish an exceedingly advantageous understanding of the nature of the transformations with which we shall have to deal, and we shall make a free use of this means of investigation.

4. Another important case of the transformation $w = az$ is that for which $\rho = 1$. The corresponding operation in the complex plane will then consist in a *rotation* of the entire plane about the origin through the angle φ. This transformation is orthogonal to that last considered; that is, every point in the plane moves in a direction at right angles to its previous motion. Every circle about the origin as a centre remains unchanged as a whole, these circles being now the lines of motion, while the rays through the origin form the orthogonal system. Each ray is converted into another at an angular distance, φ, from its initial position.

5. Let us now suppose that, in the transformation $w = az$, ρ and φ are any quantities whatever. We shall then have a combination of the two preceding operations. Every vector will be rotated about the origin through the angle φ, while its length is at the same time multiplied by ρ. The lines of motion are readily found in this case. They must be curves which have the property that, as a point moves along one of them, its radius vector, in turning through a constant angle, φ, shall be multiplied in length by a constant, ρ. The curves which have this property are logarithmic spirals having their vertices at the origin, of which the general equation is $r = Ce^{k\theta}$. For if (θ, r) and (θ', r') are the polar co-ordinates of any two points on such a spiral, and if $\theta' = \theta + \varphi$, then

$$r' = Ce^{k\theta'} = Ce^{k\theta + k\phi} = Ce^{k\theta}e^{k\phi} = e^{k\phi}r.$$

Since in the present case $r' = \rho r$, we must take $e^{k\phi} = \rho$; $\therefore k = \varphi^{-1} \log \rho$.

The lines of motion for the operation $w = az$ are therefore the logarithmic spirals $r = Ce^{\frac{\log \rho}{\phi}\theta}$. The orthogonal curves are easily found to be also logarithmic spirals, whose equations are $r = C'e^{-\frac{\phi}{\log \rho}\theta}$. Each value of the arbitrary constant C gives one spiral of the system. To determine the motion of any given point (r_0, θ_0) we must substitute these co-ordinates in the equation $r = Ce^{\frac{\log \rho}{\phi}\theta}$ and determine the corresponding value of C. We have then the equation of the particular spiral on which the given point moves.

If we assign to C successively all values from 0 to ∞, we obtain an infinite system of spirals which are coiled up one within another in such a way that through each point in the plane one, and only one, of the spirals passes. The spirals, therefore, nowhere cut each other, and they fill the entire plane. They form, therefore, a system of *congruent* curves. The concentric circles, and the origin, and the rays through the origin, are other instances of such congruent systems. And, in general, it is clear that, if a transformation has lines of motion, and if each point is transformed into a single point, the lines of motion must form a congruent system.

6. From the general logarithmic spiral motion we can easily deduce the rotation and expansion already considered, as particular cases. Thus, if $\rho = 1$, the equation $r = Ce^{\frac{\log \rho}{\phi}\theta}$ reduces to $r = C$, which gives the concentric circles. Again, if $\varphi = 0$, we have $r = \infty$, which is the only form in which the equation of a pencil of rays can be expressed in terms of r. A more satisfactory result may be got by considering the pitch of the spiral, that is the angle ψ, between the tangent and the radius vector. Since

$$\tan \psi = \frac{rd\theta}{dr},$$

we have

$$\tan \psi = \frac{\varphi}{\log \rho}.$$

Hence the pitch is 90° when $\rho = 1$, and 0 when $\varphi = 0$.

7. The student should have a clear understanding of the purpose for which the curves of motion are introduced. Any functional relation $w = f(z)$ connects each point z in the complex plane with the corresponding point w, and we are interested in the *operation* by which we may conceive that we obtain w from z. The mode of transition from the one point to the other is in no way prescribed by the functional relation. But it is convenient to regard this transition as taking place by a continuous movement of the points from the

position z to the position w along certain curves—the lines of motion. The utility of this idea lies in the fact, that, if we select the proper curve as a line of motion, then not only the one point z first considered moves along this curve, but all points on the curve are moved along on it, so that the curve as a whole remains unchanged. Moreover, the points on a line of motion move in such a way that they retain their relative order of succession, a fact which is clearly true in the cases thus far considered, and which we shall find to be true in general.

8. We consider now the theory of the *general* integral linear function $w = az + \beta$, of which the preceding cases have been special forms. This transformation may be regarded as taking place in this way: that there is first a logarithmic spiral motion about the origin defined by $z' = az$, and that this is followed by translation $w = z' + \beta$, the result being $w = az + \beta$. A simpler interpretation is, however, possible, as follows: The equation $w = az + \beta$ can also be written in the form

$$w - \beta(1 - a)^{-1} = az + \beta - \beta(1 - a)^{-1},$$

or

$$w - \beta(1 - a)^{-1} = a[z - \beta(1 - a)^{-1}].$$

The quantity $z - \beta(1 - a)^{-1}$ is evidently represented by the vector of z measured, not from the origin, but from the point $\beta(1 - a)^{-1}$; similarly, $w - \beta(1 - a)^{-1}$ is represented by the vector of w measured from the same point. If we write for these quantities z' and w' respectively, we have $w' = az'$, which evidently defines a logarithmic spiral motion about the new origin, $\beta(1 - a)^{-1}$.

The equation $w = az + \beta$ therefore defines a logarithmic spiral motion about the point $\beta(1 - a)^{-1}$. If $\beta = 0$, we have the case last considered, of rotation about the origin. If $a = 1$, we have the translation $w = z + \beta$. This may, therefore, be regarded as a limiting case of a logarithmic spiral motion in which the vertex of the spiral, $\beta(1 - a)^{-1}$, lies at an infinite distance, so that the coil of the spiral which lies in the finite part of the plane becomes a straight line. If the modulus of a be 1, or if its angle be 0, the motion becomes simply a rotation or an expansion about the point $\beta(1 - a)^{-1}$.

9. Having now obtained a geometrical interpretation of the integral linear function as defining an operation to be performed in the complex plane, we consider next more fully the effect of this operation on the parts of the plane and on their relation to each other. First of all, it is clear that the operations which we have considered are all continuous; that is, if two points are infinitely near together initially, they will remain so after the transformation.

Accordingly, any continuous curve becomes a continuous curve, and any continuous area remains such.

Again, if we draw any figure in the plane, the operation of translation will of course change only the position of the figure, leaving its shape, size, and the direction of its parts unaltered. A rotation will leave the shape and size unchanged. An expansion will leave the shape of the figure unchanged, but will change the size, and in fact will multiply each dimension of the figure by the ratio of expansion. Accordingly, the logarithmic spiral motion, being compounded from a rotation and an expansion, will also leave the shape of any portion of the plane unchanged.

Every integral linear transformation, therefore, converts every figure in the plane into a similar figure. In particular we will note, for later reference, that every straight line becomes a straight line, and every circle a circle. Again, since every figure becomes a similar figure, it follows that, if any two curves meet at an angle θ, the two corresponding transformed curves meet at the same angle; that is, every angle in the plane becomes an equal angle.

Again, the only operation thus considered which changes the size of any figure in the plane is that of expansion, and this operation evidently increases the dimensions of any part of the plane equally in all directions. It follows that, if from any point z we draw small* vectors in all directions, and consider the transformed figure, which will consist of small vectors drawn from the corresponding point w, the ratio of two corresponding small vectors will be constant on all sides of the given points. This ratio is called the ratio of similarity for the given transformation about the point z.

10. The propositions of the preceding section are also readily deduced analytically. Thus, if z and $z + dz$ be neighboring points, and w and $w + dw$ the corresponding w points, we have

$$w = az + \beta, \qquad w + dw = a(z + dz) + \beta\,; \qquad \therefore dw = adz.$$

From the last equation the continuity of the transformation is apparent. Again, from the same equation the angle of dw is equal to the angle of dz plus the constant angle of a. Accordingly, if dz be rotated about the point z, dw will rotate at the same rate about w, from which it is clear that every angle in the plane is preserved unchanged. Moreover, the ratio of the moduli of dw and dz is equal to the modulus of a, and is therefore independent of the direction in which dz is drawn from z. This ratio of similarity is, in fact, in the particular case of the integral linear functions, not only constant about

* The proposition is in this case equally true, whatever the length of the vectors may be.

every one point, but is the same for all points in the plane. In general, however, this will not be true, but different portions of the plane will be distorted in different ratios by the transformation.

11. These two properties of integral linear transformation, that every angle in the plane is converted into an equal angle and that the ratio of similarity is the same on all sides of every point, we have selected for special consideration, because these are fundamental properties of all* functions of a complex variable. Every such function defines a geometrical transformation of the complex plane of such a character that every angle is preserved and the infinitesimal region surrounding any point is equally expanded (or contracted) in all directions. That portion of the Theory of Functions which deals with the geometric representation of functional relations is therefore a part, and in fact the greater part, of the general theory of Orthomorphic Transformation.†

II. THE GENERAL LINEAR FUNCTION.

1. *The reciprocal function $w = z^{-1}$; Geometric inversion.*

12. Of the general linear function, $w = (\alpha z + \beta)(\gamma z + \delta)^{-1}$, we have thus far considered a special class, the integral linear functions, which are defined by the condition $\gamma = 0$. A second simple case occurs when $\alpha = \delta = 0$ and $\beta = \gamma$, when accordingly $w = z^{-1}$. This function z^{-1} we shall call the reciprocal function.

If we write z in its polar form, $z = r(\cos\theta + i\sin\theta)$, we have

$$w = \frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)} = \frac{1}{r}(\cos\theta - i\sin\theta) = \frac{1}{r}[\cos(-\theta) + i\sin(-\theta)].$$

The modulus of w is, therefore, r^{-1}, and its angle is $-\theta$. Geometrically the operation of passing from the point z to the corresponding point w may evidently be considered as taking place in this way: We draw a radius vector from the origin to the point z, whose polar co-ordinates we denote, as before, by r and θ, and on this radius vector we take a second point (r', θ'), such that $r' = r^{-1}$, while obviously $\theta' = \theta$. The complex number corresponding to this second point will be $z' = r^{-1}(\cos\theta + i\sin\theta)$. If now, further, we

*The term "all functions" is here to be understood as including only the functions of ordinary analysis. The statement in the text is of course true of many other classes of functions, but it is not intended here to state any proposition about "functions in general."

† "Conforme Abbildung," "Isogonale Verwandtschaft," for which Orthomorphic Transformation seems an acceptable English equivalent.

take a third point (r'', θ''), symmetrical to the second with respect to the real axis, we shall have $r'' = r' = r^{-1}$, $\theta'' = -\theta' = -\theta$; i. e. this will be the point w.

The operation implied by the equation $w = z^{-1}$ consists, therefore, of two parts. First, every point in the plane is replaced by a second point on the same radius vector through the origin as the first point, and at such a distance from the origin that $r' = r^{-1}$. This operation we shall call Geometric Inversion. Secondly, the entire resulting configuration is reflected on the real axis as if this were a mirror. This second process we shall call simply Reflection.

13. The two operations may be considered separately. The reflection on the real axis converts any portion of the plane into an equal portion symmetrical to the first with respect to the real axis. If a point moves along the boundary of the given portion in the positive direction, the corresponding point in the reflected figure will move along the reflected boundary in the negative direction. Any positive rotation will be reflected into a negative rotation. And if any angle be formed in the plane by the meeting of two curves A and B, and if the corresponding reflected curves be A' and B', then the angle *from* A *to* B will be equal and of opposite sign to the angle measured *from* A' *to* B'. Every value of z is converted into its conjugate value, which we shall denote by $\bar{z}$. Otherwise the operation of reflection is of so simple a character that it needs no special consideration. The geometric inversion is, however, not only an important theory by itself, but it is also of great importance for the further development of the present subject, and accordingly we shall devote a considerable space to its treatment.

14. To investigate the properties of the operation of geometric inversion we examine the effect of this transformation on various simple geometrical figures which we suppose drawn in the plane.* Thus, if we draw a circle of radius unity about the origin as a centre, every point on this circle will evidently be unchanged by the inversion, for, since for points on this circle $r = 1$, $r' = r^{-1} = 1$ also. This circle, which we call the unit circle, is therefore entirely unchanged by inversion. Every point which lies within the unit circle is converted into a point outside the circle, and vice versa. Any circle with its centre at the origin and lying within the unit circle will become a circle having its centre also at the origin, but lying outside the unit circle, the relation between the radii being $rr' = 1$.

Again, if any straight line be drawn through the origin, every point on

* The plane need not be regarded as the complex plane. The theory of inversion is a part of the ordinary metrical geometry.

this line will be converted into a second point on the same line. The line *as a whole*, therefore, remains unchanged, but the order of succession of its points will be altered.

If we take a point on such a line and lying very near the origin, the corresponding point will lie on the same line at a great distance, and as the first point approaches the origin the second will move off to an infinite distance. Since this is true of all lines through the origin, it appears that the origin corresponds to all the infinitely distant points of the plane, and conversely. Since, now, the infinite region of the plane is interchangeable with a single point, it is convenient to regard this infinite region as itself a single point, and we shall hereafter speak of it as the *point at infinity*.*

15. To determine what transformations figures other than straight lines through the origin and circles about the origin as a centre undergo by the operation of inversion, it is convenient to employ the rectangular and the polar co-ordinates of a point in the plane, and to obtain formulæ of transformation in terms of the former. This is very simply accomplished by the mediation of the complex quantity. Thus, if

$$z = x + yi = r(\cos\theta + i\sin\theta),$$

then

$$r = \sqrt{x^2 + y^2}, \qquad x = r\cos\theta, \qquad \text{and} \qquad y = r\sin\theta.$$

But

$$z' = \frac{1}{r}(\cos\theta + i\sin\theta) = \frac{r}{r^2}(\cos\theta + i\sin\theta) = \frac{x + yi}{x^2 + y^2}.$$

Hence

$$x' = \frac{x}{x^2 + y^2}, \qquad y' = \frac{y}{x^2 + y^2}.$$

Conversely,

$$x = \frac{x'}{x'^2 + y'^2}, \qquad \text{and} \qquad y = \frac{y'}{x'^2 + y'^2}.$$

We may now examine into what curves straight lines in the plane which do not pass through the origin are converted. The Cartesian equation of such a line is

$$Ax + By + C = 0,$$

* In general, in n-dimensional space, the infinite region is to be regarded as a linear configuration of $n - 1$ dimensions. Thus, if $n = 2$, we have the line at infinity; if $n = 3$, the plane at infinity. By n dimensions is meant n *real* dimensions. In the case of the complex number $n = 1$.

where $C \gtrless 0$. This becomes on transformation

$$\frac{Ax'}{x'^2 + y'^2} + \frac{By'}{x'^2 + y'^2} + C = 0$$

or

$$Ax' + By' + C(x'^2 + y'^2) = 0.$$

This is the equation of a circle passing through the origin. The tangent to the circle at the origin is the line $Ax + By = 0$, which is parallel to the original line, a fact which is often of use in the construction of inverted figures.

Again, the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

becomes on inversion

$$\frac{x'^2 + y'^2}{(x'^2 + y'^2)^2} + \frac{2gx' + 2fy'}{x'^2 + y'^2} + c = 0,$$

or

$$c(x'^2 + y'^2) + 2gx' + 2fy' + 1 = 0,$$

which is again a circle. An exception occurs when $c = 0$, in which case we have the straight line $2gx + 2fy + 1 = 0$.

16. To sum up the results thus far obtained, we have found that every straight line is converted into a circle passing through the origin, unless the line itself passes through the origin, in which case it remains unchanged, and that every circle becomes a circle unless it passes through the origin, in which case it becomes a straight line.

A straight line may be regarded as a circle whose centre lies at infinity, and whose radius is of infinite length. And as our transformation interchanges lines and circles, it is convenient to make no distinction between these two curves, but to regard them both as circles. We have, then, the proposition: *The operation of geometric inversion converts every circle into a circle.*

A straight line may be regarded also as a circle which passes through the point at infinity, but lies partly in the finite portion of the plane. Accordingly, since the origin was inverted into the point at infinity, any circle passing through the origin becomes a circle passing through the point at infinity, i. e. a straight line; and, conversely, any circle passing through the point at infinity, i. e. any straight line, becomes a circle through the origin.

The property that every circle becomes a circle, we shall find, is not only true of geometrical inversion, but is a fundamental characteristic of all linear transformations.

17. A second important property of inversion is deduced from the consideration of the effect of this transformation on the angle at which any two curves meet in the plane.

If A* and B be any two points in the plane, and if A' and B' be their inverse points with respect to the origin O, then $OA \,.\, OA' = OB \,.\, OB' = 1$; accordingly, $\frac{OA}{OB'} = \frac{OB}{OA'}$, and the triangles OAB and $OB'A'$ having the angle O in common are similar. The angle OAB is, therefore, equal to the angle $OB'A'$, and the angle OBA is equal to the angle $OA'B'$. Also,

$$\angle AA'B' = \angle OB'A' + \angle O = \angle A'AB + \angle O.$$

If the points A and B lie on any given curve, the points A' and B' will lie on the inverse curve. If now B approach A along the given curve, the chord AB will have for its limiting position the tangent to the curve at A, and the angle $A'AB$ will, therefore, become the angle at A between the curve and the radius vector OA. At the same time the point B' will approach the point A' along the inverse curve, the chord $A'B'$ will become a tangent to the inverse curve at A', and the angle $AA'B'$ will become the angle at A' between the tangent and the radius vector OA'. Also, since the angle at O becomes 0, we have, in the limit, $\angle AA'B' = \angle A'AB$. We have, therefore, this result: that, at the points where a radius vector through the origin cuts two inverse curves, the angle which it makes with the tangents to their curves are equal in amount, but are measured in opposite directions.

Suppose, now, that two curves BA† and CA meet at A. The corresponding inverted curves $B'A'$ and $C'A'$ will meet at A'. Then, since the angles from the radius vector to either curve and to its inverse curve are equal and opposite, their difference, i. e. the angles BAC and $B'A'C'$, will also, evidently, be equal and opposite. That is, if two curves meet at any angle, their inverse curves meet at an equal angle, but the two angles are measured in opposite directions.

The operation of geometric inversion, therefore, leaves every angle in the plane unchanged in amount, but reversed in direction. On this property and on that that every circle becomes a circle the greater part of the theory of inversion is based.

18. We have arrived at the theory of inversion from the consideration of the reciprocal function $w = z^{-1}$, and accordingly we have taken the unit circle

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* See Plate I, Fig. 1.

† See Plate I, Fig. 2.

about the origin as the fixed circle of reference. From the geometric standpoint the definition of inversion which we have thus adopted is unnecessarily restricted. Two obvious generalizations present themselves, which we shall now adopt in the geometric treatment of the subject. In the first place, it is clear that we may define a geometric inversion with respect to any centre, not necessarily the origin of the complex plane. And again, we may take as the fixed circle any circle of a radius k about the centre of inversion as a centre. Two points are then said to be inverse to each other with respect to the given circle if they lie on the same radius vector drawn from the centre of the circle at distances from the centre r and r' such that $rr' = k^2$. If then $r = k$, r' is also k, so that the given circle is really fixed with respect to the inversion as thus defined.

Two points which are inverse to each other with respect to a given circle are also called *conjugate* points with respect to this circle.

19. Given any configuration and its inverse with respect to a unit circle, if we magnify the figure in the ratio of 1 to k we shall obtain inverse configuration with respect to the circle of radius k into which the unit circle is magnified. For if r and r' are the radii vectores of two conjugate points with respect to the unit circle, and r_1 and r_1' of the corresponding points in the enlarged figure, we have $r_1 = kr$, $r_1' = kr'$. Hence $r_1 r_1' = k^2 rr' = k^2$.

It is clear, then, that the generalized operation of inversion has all the properties, with a few unimportant modifications, of the special case thus far considered. Thus every circle is converted into a circle, and every angle is preserved in amount and reversed in direction.

20. By the aid of the two fundamental properties just mentioned, we shall now deduce further properties of the operation of inversion. If a circle cut the fixed circle in two points A and B, since these points remain fixed, the inverted curve will be a second circle through the same two points. If, further, the given circle cut the fixed circle at right angles in A and B, the inverted circle will also cut the fixed circle at A and B at right angles. But since a circle can satisfy only three independent conditions,—in the present case, that it shall pass through two points in given directions,—the inverted circle and the given circle must entirely coincide.

Every circle which cuts the fixed circle at right angles is therefore unchanged as a whole by the inversion, but the order of arrangement of its points will be altered. Conversely, every circle which is unchanged by inversion cuts the fixed circle at right angles, unless it be the fixed circle itself.

If, now, we draw any radius vector through the centre of the inversion cutting any orthogonal circle in two points A and B, since the radius vector

and the circle both remain unchanged as wholes, it follows that the points A and B are interchanged by inversion. Every such radius vector, therefore, cuts out from the orthogonal circle a pair of conjugate points with respect to the circle of inversion.

Any circle through two such conjugate points will be converted into itself. For, since of two conjugate points one necessarily lies within and the other without the fixed circle, a circle through two such points must cut the fixed circle. But the operation of inversion leaves the two points of intersection unchanged, and merely interchanges the two conjugate points. Hence the given circle and its inverse circle have these four points in common, and therefore coincide throughout. The given circle is therefore converted into itself, and consequently cuts the fixed circle at right angles.

If two circles cut the fixed circle at right angles, they will intersect in two conjugate points. For, since both circles are unchanged by inversion, their two intersections must be interchanged. This property may be adopted as a definition of conjugate points, and the entire theory of inversion may be deduced from it.

21. The point conjugate to a given point with respect to a given circle is readily found by a simple geometrical construction, as follows:* If the given point lie outside the given circle, join it to the centre by a straight line. Draw the two tangents from the point to the circle, and join their points of contact by a second straight line. The intersection of these two lines is the point required.

If the given point lie within the circle, join it to the centre by a straight line. Draw a second straight line through the point perpendicular to the first. At the points where this second straight line cuts the circle draw the tangents. Their intersection is the point required.

In particular, if the given point lie at infinity, the inverse point will evidently be at the centre of the circle, and *vice versa*.

If the given circle be a straight line the conjugate of any point with respect to the line is the point obtained by reflecting the given point on the line. For, all circles which pass through the given point and cut the given line at right angles, pass also through the reflected point. It appears, therefore, that a reflection on a straight line is only a particular form of inversion.

22. We may now obtain an important proposition with regard to conjugate points. If we invert the plane with respect to any circle, conjugate points with respect to any circle become conjugate points with respect to the inverted

* See Plate I, Fig. 3.

circle. For, if we draw two or more circles through any pair of conjugate points with respect to the given circle, these circles cut the given circle at right angles. But the operation of inversion converts the given circle into a circle, and the orthogonal circles with respect to the given circle into orthogonal circles with respect to the inverted circles. Accordingly the inverted orthogonal circles pass through conjugate points with respect to the inverted circle, and therefore their two common points are conjugate to each other. But these are the inverse of the two given conjugate points.

By the aid of this proposition we can determine into what point the centre of a given circle is inverted.* For the centre is conjugate to the point at infinity with respect to the given circle. But the point at infinity becomes the centre of the circle of inversion. Consequently the centre of the given circle is inverted into the point conjugate to the centre of the circle of inversion with respect to the inverted circle.

2. *Combination of the preceding operations.*

23. We proceed, now, to a second series of developments, which are of importance for the understanding of the nature of the fractional linear functions. These developments are obtained by a combination of the operation of inversion just considered with the operations defined by the integral linear functions.

If we suppose the points in the complex plane to undergo any of the various forms of the logarithmic spiral motion, their inverse points with respect to any centre of inversion will also undergo certain systematic motions, and it is to these new species of transformation that we have next to direct our attention.

24. We begin with the case of translation. Suppose in the figure† the lines AB to be lines of motion, the lines CD belonging to the orthogonal system. The inverse of the lines AB will be a series of circles through the centre of inversion and having there a common tangent parallel to the given lines. One of the lines will pass through the centre of inversion. This line will remain unchanged, and will be the common tangent just mentioned. All the lines AB on one side of this line will become circles on one side of this common tangent, while the lines on the other side will become circles on the other side of the common tangent.

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* See Plate I, Fig. 4.

† See Plate II, Fig. 5.

The lines CD will become an equal system of circles, orthogonal to the first system and having for their common tangent at the centre of inversion that line CD which passes through the origin.

If now a point moves along one of the lines, the inverse point will move along the corresponding circle. A motion of the point along the entire length of the line AB from infinity to infinity will correspond to a motion of the inverted point from the centre of inversion entirely around the corresponding circle back to the starting point. To a translation of the given plane corresponds, therefore, a deformation of the inverted plane, in which every point moves along that circle which passes through it and corresponds to a line of motion of the translation. This species of motion is called the Parabolic Motion. The corresponding directions of motion are indicated in the figure by arrow heads.

25. A second important class of new motions is obtained in the same way by inverting the motion of rotation in the complex plane.* If the point P be the centre of such a motion, the lines of motion are circles about P as a centre, while the orthogonal system is composed of straight lines through P. If we invert the plane, the point P becomes the inverse point P'. The circles about P whose radii are less than OP will become circles surrounding P'. They will be symmetrical with respect to the line OP, but will not, however, have their centres at P'. They are so situated that, if a straight line be drawn through P' perpendicular to OP, the pairs of tangents drawn at the points where this line cuts the successive circles will meet at the centre of inversion O.

The circle about P which passes through the origin will become a straight line perpendicular to OP and bisecting OP'. And if the radius of the original circle about P be greater than OP, i. e. if this circle include the origin, the inverted circle will also include the origin. Moreover P' will be conjugate to the origin with respect to the inverted circle, and consequently, will lie outside of it. And if tangents be drawn to the inverted circle from P', their chord of contact will pass through O. The circles about P whose radii are greater than OP will, therefore, invert into a system of circles surrounding the origin, arranged in the same way as the circles about P'.

Of the orthogonal system, the line OP will be unchanged, and is the axis of symmetry of the configuration. Every other line through P will become a circle through O and P', and this system of circles will be orthogonal to the first system.

If now a rotation take place in the complex plane about the point P, all

* See Plate II, Fig. 6.

points in the plane moving along the circles about P, the inverse points will move along the circles about P' and O, the direction of motion being indicated by the arrow heads. This motion is called the Double Circular, or Elliptic Motion.

Again, if we suppose an expansion to take place from P outward, all points in the plane moving along the system of rays through P, the inverse points will move along the system of circles through O and P'. In particular, if a point start from P and move along the ray through P to infinity, the inverse point will start at P' and move to O, along the corresponding circle. This species of motion is called the Asymptotic, or Hyperbolic, motion.

26. If we combine the motions of expansion and rotation about the point P, thus producing the logarithmic spiral motion, the result in the inverted plane will be a combination of the elliptic and hyperbolic motions. The inverted points will move about P', while at the same time they are carried away from P' toward O. The curves of motion are easily seen to be Double Spirals* having the points P' and O as poles.

27. In all the forms of transformation which we have considered, there will pass through every point of the plane one, and only one, line of motion. All points on such a line will be moved along this line in such a way that their order of succession, and that the connections of the parts of the plane in general, are unchanged. A clear understanding of this principle lies at the foundation of the entire present theory.

28. The equation of the double spiral, in so-called bipolar co-ordinates, may be obtained by the following method, which is one of considerable application: Let AB† be an arc of the single spiral with its pole at P, $A'B'$ the corresponding arc of the double spiral with one pole at P', the inverse of P, and the other at the origin O. Denote OA' by r_2, $P'A'$ by r_1, the angle $P'OA'$ by φ, the angle $OP'A'$ by ψ, and the angle $AA'P'$ by χ. The equation of the single spiral is $r = Ce^{k\theta}$. But

$$\frac{r_1}{r_2} = \frac{r}{OP}, \quad \text{and} \quad \theta = \chi = \varphi + \psi ;$$

hence

$$\frac{r_1}{r_2} = C'e^{k(\phi+\psi)} = C'e^{k\chi}.$$

So long as the double spiral coils about P', we have, whenever it crosses OP, $\varphi = 0$, $\psi = k\pi$. Therefore, at each passage around P' the ratio $r_1 r_2^{-1}$

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* See Plate II, Fig. 7.

† See Plate I, Fig. 8.

is multiplied by $e^{2k\pi}$. Consequently the successive coils lie further and further from P'. Finally, the double spiral will have a point of inflection at its intersection with OP, and from that point it will begin to coil about O. At each intersection with OP we shall then have $\psi = 2n\pi$, where n is a fixed integer, while $\varphi = m\pi$, where m is the number of turns performed about O. Consequently $\chi = (2n + m)\pi$, so that the ratio $r_1 r_2^{-1}$ will still increase, and r_2 will approach the value 0.

The orthogonal system of curves are also double spirals, and their equations are $r_1 r_2^{-1} = Ce^{-k^{-1}\chi}$. The constant k determines the pitch of the original logarithmic spirals, while the constants C are determined so soon as any point is specified through which the corresponding spiral is to pass.

The reader will easily deduce analytically from the general formula the special cases of the Elliptic and Hyperbolic motion.

29. The linear transformation $w = z^{-1}$, the consideration of which was the starting point of the preceding developments, consisted of two parts, an inversion of the plane with respect to the unit circle about the origin and a reflection of the plane on the real axis. The former operation we have extended, for the sake of generality, to include inversion with respect to any circles whatever. We may likewise take into consideration the reflections of the plane on any straight line. These reflections in themselves are obviously very simple. Considerable interest attaches, however, to the combinations of reflections with inversions. One important property of such a compound operation is at once clear. Since every inversion and every reflection preserves every angle in the plane in amount but reverses it in direction, it follows that a combination of an inversion with a reflection, in particular the reciprocation, $w = z^{-1}$, leaves every angle unchanged both in amount and direction.

30. If we suppose the system of curves which we have just constructed from logarithmic spirals by the aid of inversion to be subjected further to a reflection on any straight line, the character of these curves will evidently be entirely unchanged. They will still be double spirals of various forms as before. Indeed, if we have already constructed all possible double spirals, the reflection will merely interchange those on one side of the fixed line with those on the other side. One important effect of this rearrangement must, however, be noticed. If P be the pole of a logarithmic spiral motion, and if P' be the corresponding pole of the inverted motion, then if a point move about P on the given spiral in either direction, the corresponding inverted point will move about P' along the inverted curve in the opposite direction. Whereas, in the case of the inverted and reflected curve, corresponding points move about corresponding poles in the same directions.

3. *Connection of the double spirals with the theory of the linear functions of the complex variable.*

31. The further general theory of combinations of inversions and reflections we postpone to a later section. At present we are passing gradually to the theory of linear transformation, and it is convenient to employ only the operation of reciprocation, because this is already defined as a linear function. We shall, therefore, restrict ourselves to inversion with respect to the unit circle about the origin and reflection on the real axis. This restriction is, however, rather apparent than real; for if we have to invert any configuration with respect to a circle of radius k, the relation between the radii of corresponding points is $r' = k^2 r^{-1}$. But if we suppose the plane, as given, to contract towards the centre of inversion in the ratio of $k^2 : 1$, the new radius vector of any point is $r_1 = rk^{-2}$. Now, if we invert the plane with respect to the unit circle about the centre of inversion, we have $r' = r_1^{-1} = k^2 r^{-1}$, as before. But the contraction of the plane converts every figure into a similar figure. Consequently, it appears that every curve that can be obtained by inverting logarithmic spirals with respect to a circle of radius k, can also be obtained by inverting logarithmic spirals with respect to a circle of radius 1.

Again, if the unit circle of inversion does not have its centre at the origin, we may translate the entire plane until the centre of the given circle shall coincide with the origin. Then we may invert it with respect to the translated circle, and finally retranslate it until the circle of inversion reassume its initial position. The result will obviously be the same as though the plane had simply been inverted with respect to the given circle. It is clear, then, that every double spiral can be obtained from a logarithmic spiral by a translation of the plane, an inversion with respect to the unit circle about the origin, and a second translation equal and opposite to the first. For the sake of analytic simplicity we now suppose that in the series of operations the inversion is replaced by the reciprocation $w = z^{-1}$. We may then state at once the proposition, that every double spiral can be obtained from a logarithmic spiral by a translation, a reciprocation, and a second translation equal and opposite to the first.

32. For the present purpose the interest in the system of double spiral curves, including the Elliptic, Parabolic, and Hyperbolic cases, centres in the fact that these systems of curves are the lines of motion corresponding to linear functions of a complex variable, and, conversely, that every linear function of a complex variable defines a geometrical transformation of the types

which we have constructed. It is the establishment of this proposition which constitutes the next important development of the subject.

It was simply for convenience in the examination of this identity of our geometric processes with the linear transformations that we suppose the system of curves which we had obtained by inversion to undergo a reflection also. These curves, whether reflected or not, are the lines of motion of linear transformation, but the process by which we pass from a given curve to its inverse curve is not a linear transformation of a complex variable, whereas if we add the reflection the operation becomes a pair of translations and a reciprocation, all of which are linear transformations.

33. We proceed now actually to determine for each of the forms of motion which we have constructed the corresponding analytic transformation of the complex variable. We begin with the simple case of the parabolic motion about the origin as a pole.

If z and w be the complex numbers corresponding to the initial and final position of any point in the plane which is to undergo the given motion, we are to obtain the analytic definition of this motion in the form of an equation connecting w and z. We will denote the points reciprocal to w and z by z_2 and z_1, respectively, so that $z = z_1^{-1}$ and $w = z_2^{-1}$. Now, when the point z undergoes the parabolic motion, passing from the position z along the corresponding circle to the position w, the reciprocal point z_1 undergoes a translation β to the position z_2, so that $z_2 = z_1 + \beta$. We have, then, $w = z_2^{-1}$, $z_2 = z_1 + \beta$, $z_1 = z^{-1}$, and from these we obtain at once $w = z(1 + \beta z)^{-1}$. w is, therefore, as we have asserted, a linear function of z.

The parabolic motion about the origin as a pole is therefore defined by the linear equation $w = z(1 + \beta z)^{-1}$, where the constant β indicates the amount and direction of the corresponding translation of the reciprocated plane.

If the pole of the motion lie at a point γ instead of at the origin, we can bring this point to the origin by the translation $z_1 = z - \gamma$. When z performs the given parabolic motion, z_1 will perform an equal parabolic motion about the origin. Accordingly, if z_2 be the final position of z_1, we have from the above $z_2 = z_1(1 + \beta z_1)^{-1}$. Now the final position of z_1, which we will denote by w, is $z_2 + \gamma$. So that we have

$$w = z_2 + \gamma = \frac{z_1}{1 + \beta z_1} + \gamma = \frac{z - \gamma}{1 + \beta(z - \gamma)} + \gamma = \frac{(1 + \beta\gamma)z - \beta\gamma^2}{\beta z + 1 - \beta\gamma},$$

which is again a linear transformation.

A parabolic motion about any point γ in the complex plane is therefore defined by an equation

$$w = \frac{(1 + \beta\gamma) z - \beta\gamma^2}{\beta z + 1 - \beta\gamma},$$

where β defines the translation reciprocal to the given motion with respect to its pole.

34. A separate consideration of the elliptic and hyperbolic motions is unnecessary. We proceed at once to the treatment of their combination, the double spiral motion, and of this motion we consider immediately the general form, where the poles of the double spiral are any points γ and δ in the plane. The pole γ may be brought to the origin by the translation $z_1 = z - \gamma$; then the point reciprocal to z_1, $z_2 = z_1^{-1}$, will perform a logarithmic spiral motion defined by the equation $z_3 = az_2 + \beta$. The final position of the point z_1 will be the reciprocal point to z_3, $z_4 = z_3^{-1}$, and the final position of z will be $w = z_4 + \gamma$. Hence,

$$w = z_4 + \gamma = \frac{1}{z_3} + \gamma = \frac{1}{az_2 + \beta} + \gamma = \frac{z_1}{a + \beta z_1} + \gamma$$
$$= \frac{z - \gamma}{a + \beta(z - \gamma)} + \gamma = \frac{(1 + \beta\gamma) z - \gamma + a\gamma - \beta\gamma^2}{\beta z + a - \beta\gamma},$$

which is again a linear function.

This function can be simplified by the introduction of the second pole δ of the spiral in the place of β. This can be accomplished as follows: The translation $z_1 = z - \gamma$ converts the point δ into $\delta - \gamma$; the single spiral motion $z_3 = az_2 + \beta$ takes place about the reciprocal of this point. Hence,

$$\frac{\beta}{1 - a} = \frac{1}{\delta - \gamma}, \quad \text{or} \quad \beta = \frac{1 - a}{\delta - \gamma}.$$

From this we readily obtain

$$w = \frac{\left[1 + \frac{(1 - a)\gamma}{\delta - \gamma}\right] z - \gamma + a\gamma - \frac{1 - a}{\delta - \gamma}\gamma^2}{\frac{1 - a}{\delta - \gamma} z + a - \frac{1 - a}{\delta - \gamma}\gamma} = \frac{(\delta - a\gamma) z + \gamma\delta (a - 1)}{(1 - a) z + a\delta - \gamma}.$$

The constant a in this formula defines the extent of the single spiral motion which is the reciprocal of the given double spiral motion with respect to the pole γ. If a be real, the equation defines an hyperbolic motion. If the modulus of a be 1, the equation defines an elliptic motion.

35. The equation between w and z may also be written in the form

$$w = \frac{\delta (z - \gamma) - a\gamma (z - \delta)}{z - \gamma - a (z - \delta)},$$

in which the relation of the motion to the two poles of the double spiral is more apparent. These two poles γ and δ do not enter symmetrically into the equation; but if we interchange γ and δ, and at the same time write for a a^{-1}, the equation remains unchanged.

The explanation of these facts is readily obtained from geometrical considerations. In the double spiral motion, the motion about the two poles takes place in opposite directions, so that a complete symmetry of our equation with respect to γ and δ is not to be expected. The reciprocal of the double spiral motion with respect to the pole γ is the logarithmic spiral motion $z' = az$, if we omit the constant term β, which does not affect the character of the motion, but only serves to fix the position of the pole. Since now, when we interchange γ and δ, we must also put for a a^{-1}, it appears that the reciprocal of the double spiral motion with respect to the pole δ is the logarithmic spiral motion $z' = a^{-1}z$, omitting again the constant β'.

These two logarithmic spiral motions $z' = az$ and $z' = a^{-1}z$ are evidently exactly the reverse of each other. They take place along the same system of curves, but if one of them carries a point z to z', the other will carry z' back over the same path to z. Accordingly, the motion about the two poles of the double spirals would be equal and opposite along the same system of curves if the two poles were brought into coincidence. To state the result in a slightly different form, in which the symmetry of the motion about the two poles is clearly exhibited, suppose we construct two equal and coincident complete systems of double spirals about any two points as poles. Then, if we hold the one system fixed, and turn the other bodily about until the positions of its poles are interchanged, the second system will again exactly coincide with the first. It must, however, be noted that this symmetry is true only of the entire system of curves, but not of the individual spirals. These are not symmetrical about the two poles, but when their coils about the one pole are large, those about the other pole are correspondingly small. All that is shown is that any one of the spirals on having its poles interchanged will become equal to a second spiral.

36. It appears from the above considerations that when we interchange the two poles of the motion we obtain a second motion equal and opposite to the first. The two analytic transformations

$$w = \frac{(\delta - a\gamma) z + \gamma\delta (a - 1)}{(1 - a) z + a\delta - \gamma} \quad \text{and} \quad w = \frac{(\gamma - a\delta) z + \gamma\delta (a - 1)}{(1 - a) z + a\gamma - \delta}$$

should, therefore, be reverse to each other; that is, if the former converts z into w, the latter should convert w into z. That this is the case is seen by solving the first equation for z. We have

$$z = \frac{(\gamma - \alpha\delta)\, w + \gamma\delta\,(\alpha - 1)}{(1 - \alpha)\, w + \alpha\gamma - \delta},$$

which is exactly the second transformation.

A third and even simpler form of the equation between w and z, which we shall frequently find of use, may be deduced synthetically as follows: When $z = \delta$, w is also δ; hence $w - \delta$ must contain $z - \delta$ as a factor. Similarly, $w - \gamma$ contains $z - \gamma$ as a factor; hence $(w - \gamma)(w - \delta)^{-1}$ contains $(z - \gamma)(z - \delta)^{-1}$ as a factor. But $(w - \gamma)(w - \delta)^{-1}$ is a linear function of z, and can therefore differ from $(z - \gamma)(z - \delta)^{-1}$ only by a constant factor. This factor is readily found to be α. So that we have

$$\frac{w - \gamma}{w - \delta} = \alpha\,\frac{z - \gamma}{z - \delta}.$$

4. *The interpretation of the linear transformations.*

37. Having now shown that all the systems of curves which we have considered are lines of motion corresponding to linear transformations, we proceed next to establish the converse proposition, that all linear transformations have for their lines of motion the systems of curves which we have considered.

For this purpose, we begin by enquiring whether a linear transformation $w = (\alpha z + \beta)(\gamma z + \delta)^{-1}$ leaves any points in the plane unchanged in position. If this be the case, we have for these points $w = z$; hence $z = (\alpha z + \beta)(\gamma z + \delta)^{-1}$. This reduces to $\gamma z^2 + (\delta - \alpha)\, z - \beta = 0$, which is a quadratic equation, and accordingly has two roots. We have, therefore, the general proposition, that every linear transformation leaves two points in the complex plane fixed.

These two points may be finite and distinct, finite and coincident, or one may be finite while the other is infinite, or finally they may both be infinite. The condition that one of the roots of the quadratic equation should be infinite is that the coefficient of z^2 shall vanish; that is, $\gamma = 0$. If this be the case, the linear transformation reduces to the integral transformation $w = \alpha\delta^{-1}\gamma + \beta\delta^{-1}$, which defines a logarithmic spiral motion in the plane about the point $\beta\,(\delta - \alpha)^{-1}$, which is the other solution of the quadratic. The logarithmic spiral motion therefore, like the double spiral motion, has

two fixed points, or *poles*, one of which is the vertex of the spiral, while the other lies at infinity. Geometrically this is also clear. Conversely, every linear transformation which has one infinite and one finite fixed point is a logarithmic spiral motion.

38. The second fixed point will also lie at infinity if the coefficient of z in the quadratic equation vanishes also, i. e. if $\delta = \alpha$. The linear transformation then reduces to the form $w = z + \beta\alpha^{-1}$, which is a translation. A translation is therefore to be regarded as a special case of linear transformation for which the two fixed points coincide and lie at infinity. Conversely, every linear transformation which possesses this property is a translation.

39. There remain to be considered, 1° the case of finite coincident points, and 2° the case of finite distinct points.

In the first of these two cases suppose the double fixed point to lie at η. If in the equation of the given transformation we write for z, $z' + \eta$, and for w, $w' + \eta$, we shall have a new linear transformation. Moreover, when $z = \eta$, w is also equal to η. Consequently, when $z' = 0$, w' is also 0, and there is no other value of z' for which w' has the same value. The second linear transformation has, therefore, for its fixed points the double point 0. Geometrically the operations $z = z' + \eta$ and $w = w' + \eta$ define a translation which carries the point η to the origin and moves the system of lines of motion bodily with it without changing them otherwise. Again, if we write in the second transformation $z' = z''^{-1}$, $w' = w''^{-1}$, we have still a linear transformation with a double fixed point at infinity. But every linear transformation which has this property is a translation. The second transformation, which is the reciprocal of this, is therefore a parabolic motion about the origin; and consequently the original motion, from which the second was obtained by the translation $z' = z - \eta$, is also a parabolic motion about the point η.

Every linear transformation which leaves only one point in the plane fixed is, therefore, a parabolic motion about this point as a pole. In particular, if the fixed point lie at infinity, the motion is a translation.

40. If the two poles of the linear transformation be finite and distinct, we may bring one of them to the origin by a translation of the plane. By reciprocation with respect to the origin, this point becomes the point at infinity, while the second pole is converted by these two operations into a second finite point. Both the operations of translation and reciprocation convert a linear function into a linear function. The resulting transformation is therefore a linear transformation with one fixed point at infinity. But every such linear transformation is a logarithmic spiral motion. Reversing our auxiliary operations, we have first the reciprocal of the logarithmic spiral motion. But this

is a double spiral motion about the origin and the point into which the second pole of the original transformation was translated. Again reversing the translation, we have finally a double spiral motion about the two fixed points as poles.

Every linear transformation, therefore, which leaves two finite and distinct points unchanged is a double spiral motion, which may in special cases reduce to a hyperbolic or an elliptic motion.

41. We have now considered every possible case, and may state the result as follows: Every linear transformation leaves either two distinct or two coincident points in the plane fixed. In the former case the transformation is a double spiral motion, which may reduce in special cases to a hyperbolic or an elliptic motion, or if one pole lie at infinity, to a logarithmic spiral motion, which again includes the operations of rotation and expansion as particular forms. In the latter case the transformation is a parabolic motion about the fixed point, which reduces to a translation, if the double fixed point lie at infinity. The parabolic motion again may be regarded as a special form of the double spiral motion for which the two poles coincide. Accordingly we have the final result: Every linear transformation is identical with some one of the various forms of a double spiral motion.

42. To complete this portion of the theory of the linear transformation, it remains to obtain formulæ for the fixed points of a transformation in terms of its coefficients, and to establish an analytical criterion for the cases where the double spiral motion reduces to the parabolic, elliptic, or hyperbolic form.

The fixed points of a linear transformation we shall hereafter call its foci. They are obtained from the equation

$$\gamma^2 + (\delta - \alpha) z - \beta = 0,$$

and are

$$z_1 = \frac{\alpha - \delta + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}}{2\gamma} \quad \text{and} \quad z_2 = \frac{\alpha - \delta - \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}}{2\gamma}.$$

From the equation we have also

$$z_1 + z_2 = \frac{\alpha - \delta}{\gamma} \quad \text{and} \quad z_1 z_2 = -\frac{\beta}{\gamma},$$

formulæ which are often useful.

In the particular case of the parabolic motion we have, further,

$$(\alpha - \delta)^2 + 4\beta\gamma = 0, \quad z_1 = z_2 = \frac{\alpha - \delta}{2\gamma}.$$

The criteria for the elliptic and hyperbolic cases are readily determined by the aid of the equation of the double spiral motion in terms of its poles and of the quantity α of Sec. 34, for which we will now write ε,

$$w = \frac{(z_2 - \varepsilon z_1) z + z_1 z_2 (\varepsilon - 1)}{(1 - \varepsilon) z + \varepsilon z_2 - z_1}.$$

If this be identical with the transformation $w = (\alpha z + \beta)(\gamma z + \delta)^{-1}$, we must have

$$\begin{aligned} z_2 - \varepsilon z_1 &= k\alpha, & 1 - \varepsilon &= k\gamma, \\ z_1 z_2 (\varepsilon - 1) &= k\beta, & \varepsilon z_2 - z_1 &= k\delta; \end{aligned}$$

where k is an arbitrary constant.

$$\therefore \frac{z_2 - \varepsilon z_1}{\varepsilon z_2 - z_1} = \frac{\alpha}{\delta},$$

$$\varepsilon = \frac{z_2\delta + z_1\alpha}{z_2\alpha + z_1\delta} = \frac{\alpha^2 - \delta^2 + (\alpha - \delta)\sqrt{(\alpha-\delta)^2 + 4\beta\gamma}}{\alpha^2 - \delta^2 - (\alpha - \delta)\sqrt{(\alpha-\delta)^2 + 4\beta\gamma}} = \frac{\alpha + \delta + \sqrt{(\alpha-\delta)^2 + 4\beta\gamma}}{\alpha + \delta - \sqrt{(\alpha-\delta)^2 + 4\beta\gamma}},$$

unless $\alpha = \delta$. If $\alpha = \delta$, $z_1 = -z_2$, and we have

$$\frac{\alpha}{\beta} = \frac{z_2 - \varepsilon z_1}{z_1 z_2 (\varepsilon - 1)} = \frac{1 + \varepsilon}{z_1 (1 - \varepsilon)};$$

$$\therefore \varepsilon = \frac{-\alpha z_1 - \beta}{\beta - z_1 \alpha} = \frac{-\alpha\sqrt{\beta} - \beta\sqrt{\gamma}}{\beta\sqrt{\gamma} - \alpha\sqrt{\beta}} = \frac{\alpha + \sqrt{\beta\gamma}}{\alpha - \sqrt{\beta\gamma}}.$$

The general formula, however, reduces to this same quantity, if $\alpha = \delta$. The conditions for elliptic or hyperbolic motion are, therefore, that

$$\frac{\alpha + \delta + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}}{\alpha + \delta - \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}} \quad \text{or} \quad \frac{[\alpha + \delta + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}]^2}{4(\alpha\delta - \beta\gamma)}$$

shall have its modulus equal to 1, or its angle equal to 0, respectively.

43. We can simplify this condition by the following considerations: Given a linear transformation $w = (\alpha z + \beta)(\gamma z + \delta)^{-1}$, the quantity $\alpha\delta - \beta\gamma$ is called the *determinant* of the transformation. This determinant is never 0, for then we should have $\alpha\gamma^{-1} = \beta\delta^{-1}$, and the given transformation would accordingly reduce to $w = \alpha\gamma^{-1}$, and would therefore cease to be a proper transformation.

Now we may divide or multiply all the coefficients α, β, γ, and δ in the transformation by any same number whatever, without altering the transformation itself in any way. We will make use of this fact, and will divide each of the coefficients by $\sqrt{\alpha\delta - \beta\gamma}$, calling the resulting quotients α', β', γ', and δ'. The determinant of the transformation in its new form is, then,

$$\alpha'\delta' - \beta'\gamma' = \frac{\alpha\delta}{\alpha\delta - \beta\gamma} - \frac{\beta\gamma}{\alpha\delta - \beta\gamma} = 1.$$

We may therefore always suppose any linear transformation to be given in such a form that its determinant is equal to unity. This form we will call the *normal form.*

For the normal form the quantity

$$\frac{[\alpha + \delta + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}]^2}{4(\alpha\delta - \beta\gamma)},$$

which can also be written

$$\frac{[\frac{1}{2}(\alpha + \delta) + \sqrt{\frac{1}{2}(\alpha + \delta)^2 + \beta\gamma - \alpha\delta}]^2}{\alpha\delta - \beta\gamma},$$

reduces to

$$[\tfrac{1}{2}(\alpha + \delta) + \sqrt{\tfrac{1}{2}(\alpha + \delta)^2 - 1}]^2.$$

If, now, we write $\frac{1}{2}(\alpha + \delta) = \cos\varphi$ where φ is some complex number, we have at once

$$\varepsilon = (\cos\varphi + i\sin\varphi)^2 = \cos(2\varphi) + i\sin(2\varphi).$$

But if the modulus of ε be 1, then $\varepsilon = \cos\psi + i\sin\psi$, where ψ is a *real* angle. Hence, if the given transformation be an elliptic motion, 2φ, and consequently φ, must be a real angle. The sufficient and necessary condition, therefore, that a linear transformation, given in its normal form, shall be elliptic is, that $\frac{1}{2}(\alpha + \delta) = \cos\varphi$, where φ is a real angle, i. e. that $\frac{1}{2}(\alpha + \delta)$ shall be real and less than 1. The angle φ is then half the angle of the corresponding rotation. The condition that the transformation shall be hyperbolic is, that ε or $\frac{1}{2}(\alpha + \delta) + \sqrt{\frac{1}{2}(\alpha + \delta)^2 - 1}$ shall be real. This can only happen when $\frac{1}{2}(\alpha + \delta)$ is real and greater than 1. The conditions for elliptic, parabolic, and hyperbolic transformations are, therefore, for the normal form, that $\frac{1}{2}(\alpha + \delta)$ shall be real, and < 1, $= 1$, or > 1, respectively. The corresponding conditions for the case of linear transformations not in the normal form are, that $\frac{1}{2}(\alpha + \delta)(\alpha\delta - \beta\gamma)^{-\frac{1}{2}}$ shall be real, and < 1, $= 1$, or > 1, respectively.

44. We examine next the number of conditions which suffice to determine the various forms of the linear transformation. The general transformation $w = (\alpha z + \beta)(\gamma z + \delta)^{-1}$ contains 4 arbitrary constants, α, β, γ, δ. Since, however, only their ratios are of account, these reduce to 3 essential constants. Each of them is composed of two independent constants, its real part and its imaginary part. A general linear transformation depends, therefore, on 6 arbitrary constants, and accordingly can be subjected to 6 independent conditions. Each of the 6 constants can vary from $-\infty$ to $+\infty$. There is, therefore, a sixfold infinite number of linear transformations of a complex variable. We state this now briefly by saying that there are ∞^6 such transformations.

We may arrive at the same result by other considerations. Thus the general linear transformation is a double spiral motion. This is determined as soon as we know its poles and the constant ε. Each pole is determined by its two co-ordinates, and ε involves two constants, making 6 in all.

The elliptic and hyperbolic motion are each subject to one condition, that the modulus of ε shall be 1, or that its angle shall be 0. These motions are therefore determined by 5 further conditions. If their two poles be given, these are equivalent to 4 constants. The remaining constant determines the *extent* of the motion. It is to be noted that if the two poles of an elliptic or hyperbolic motion are given, the entire system of lines of motion is fixed. For the lines of motion in the former case are circles surrounding the two poles in such a way that, if the poles be joined by a straight line and a perpendicular erected to this at one of the poles, the tangents to each circle of motion at the points where it meets this perpendicular meet at the other pole. Such a system of circles can clearly be constructed only in one way when the poles are once given, and the same is therefore true of the orthogonal system. Accordingly, when the poles are given, it only remains to fix the *extent* of the motion along the known circles; whereas in the case of the true double spiral motion, when the two poles are given, we can still construct ∞^1 systems of spirals which shall have these poles. We must still have, therefore, one constant to determine the particular system of spirals, and one to determine the extent of the motion on this particular system.

The parabolic motion is defined by the equation $(\alpha - \delta)^2 + 4\beta\gamma = 0$, which, since the real parts and the imaginary parts must be equal, is equivalent to 2 conditions. It requires, therefore, 4 more conditions to determine a particular parabolic motion. Thus, if the pole of the motion be given, the common tangent to the circles of motion at the pole, and the extent of the motion, these are equivalent to 2, 1, 1 conditions, respectively, and determine the motion.

The reader can easily extend these considerations to the case of the single spiral motion.

45. Having now studied the theory of the lines of motion corresponding to linear transformations of a complex variable, we have further to consider, as in the case of the integral linear function, the effect of these transformations in distorting the different portions of the plane and in altering their relations to each other.

All linear transformations, as we have seen, are obtainable from the various forms of the logarithmic spiral motion by the aid of the operations of translation and reciprocation. Each of them has by itself the properties that every continuous series of points in the plane remains a continuous series of points, that every angle is converted into an equal angle, and, we will also note, that every circle becomes a circle. The logarithmic spiral motion and the translation have the further property, that the ratio of similarity about every point is the same on all sides of the point. A brief consideration will show that this is also true of the reciprocation. All combinations of these operations possess obviously these same properties. It appears, then, that every linear transformation is continuous, leaves every angle unchanged, and distorts the elementary region surrounding any point equally in all directions, while it also converts every circle into a circle.*

Analytically, if z and $z + dz$ are two neighboring points, and w and $w + dw$ the corresponding transformed points, we have

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad w + dw = \frac{\alpha(z + dz) + \beta}{\gamma(z + dz) + \delta}.$$

Hence,

$$dw = \frac{(\alpha\delta - \beta\gamma)\,dz}{(\gamma z + \delta)\,[\gamma(z + dz) + \delta]}.$$

From this equation the continuity of the transformation is apparent. Moreover, we have, in the limit, $dw/dz = (\alpha\delta - \beta\gamma)(\gamma z + \delta)^{-2}$, and since the quantity on the right-hand side of this equation is independent of dz, the ratio of similarity at any point z is independent of the direction of dz, i. e. it is the same on all sides of z. And from the same equation it is clear that every angle is preserved.

It will be noted that the ratio of similarity is no longer, as in the case of the integral linear functions, constant throughout the plane, but is a variable

* We may add that conjugate points with respect to any circle become conjugate points with respect to the transformed circle.

quantity, so that different parts of the plane are distorted in entirely different ratios.

We can also obtain a clear conception of the effect of a linear transformation in distorting the plane as follows: Construct any two of the lines of motion of the transformation. These include between them a certain area. Every such area is converted by the transformation in itself. If, in addition, we construct any two curves of the orthogonal system, these together with the two lines of motion include a curvilinear quadrilateral. The transformation will move this quadrilateral along between the two lines of motion and will leave it still bounded by two orthogonal curves. By drawing the orthogonal curves at proper intervals we can easily arrange, if we wish, that each quadrilateral shall be converted exactly into the next succeeding quadrilateral between the same pair of lines of motion.

46. In the further extension of the subject, the elliptic motions are of especial importance, and accordingly we consider next a few examples of this type from the point of view of the preceding developments.

We begin with the transformation $w = z^{-1}$, which we have already treated from a different stand-point. That this is an elliptic motion appears at once from the formula in Sec. 42, which gives us $\varepsilon = -1$, so that the corresponding logarithmic spiral motion in this case reduces to a rotation through 180°.

The foci of the transformation are evidently the points $+1$ and -1. The lines of motion are circles surrounding these two points. The orthogonal system is composed of circles passing through the two points. These two systems of curves correspond respectively to the circles about the centre of the rotation through 180° and to the rays through this point. In the case of this rotation there passes through every point in the plane one circle of motion and one orthogonal ray, and the effect of the rotation is to carry the point along the circle to the second intersection of the circle with the ray. Similarly, in the case of the transformation $w = z^{-1}$, there passes one circle of motion and one orthogonal circle through every point in the plane, and the effect of the transformation is to carry the point along the circle of motion to the second intersection of this circle with the orthogonal circle. A repetition of this transformation evidently restores every point in the plane to its initial position.

The geometrical character of the transformation will now be clear from the figure,* in which the curvilinear quadrilaterals, bounded in each case by

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* Plate III, Fig. 9.

two circles of motion and two orthogonal circles, are so lettered that each pair of quadrilaterals which are converted one into the other by the transformation are marked by the same letter.* The continuity of the transformation comes clearly to view, and the property is also geometrically apparent that the transformation, if repeated, restores every point to its original position. Analytically, it is clear that a repetition of the transformation $w=z^{-1}$ must restore every point in the plane to its initial position; for if $w=z^{-1}$ and $w'=w^{-1}$, then $w'=z$. A transformation which has the property that, on being repeated n times, it restores every point in the plane to its initial position is said to be periodic of period n. The transformation $w=z^{-1}$ is therefore periodic of period 2. Such a transformation is called an involution. The theory of periodic transformations will play a highly important part in the later developments of the subject.

47. As a second example we consider the transformation $w=(z+i)(iz+1)^{-1}$. From the formula of Sec. 42 we have in this case $\varepsilon=i$, so that the corresponding logarithmic spiral motion reduces to a rotation through an angle of 90°. The given transformation is therefore an elliptic motion, evidently of period 4.

The foci of the transformations are the solutions of the equations $z=(z+i)(iz+1)^{-1}$ or $iz^2=i$, whence $z=+1$ or -1. The lines of motion and the orthogonal system are, therefore, the same as in the case of the transformation $w=z^{-1}$.† But if a point lie at the intersection of a given circle of motion and a given orthogonal circle, it will be moved along the circle of motion until it meets the orthogonal circle which cuts the given orthogonal circle at right angles. A repetition of the operation must therefore be equivalent to the transformation $w=z^{-1}$. This is readily verified analytically. If $w=(z+i)(iz+1)^{-1}$ and $w'=(w+i)(iw+1)^{-1}$, then

$$w'=\frac{\dfrac{z+i}{iz+1}+i}{i\dfrac{z+i}{iz+1}+1}=\frac{2i}{2iz}=\frac{1}{z}.$$

A second repetition of the transformation gives

$$w''=\frac{w'+i}{iw'+1}=\frac{1+iz}{i+z},$$

* Quadrilaterals which are symmetrical to each other with respect to the imaginary axis are also marked alike, but this will lead to no confusion.

† Since the foci are the same. Cf. Sec. 44.

while a third repetition gives

$$w''' = \frac{w'' + i}{iw'' + 1} = \frac{2iz}{2i} = z\,;$$

that is, every point is restored to its original position, and the period of the transformation is evidently 4. In the accompanying figure* the quadrilaterals which are converted into one another by the repetitions of the transformation are all marked with the same letter, so that each letter occurs four times about each focus.

The transformations which arise from the repetition of a given transformation we shall hereafter call the *powers* of the transformation. Thus in the present case, if we denote the transformation $w = (z + i)(iz + 1)^{-1}$ by S, the transformations $w = z^{-1}$, $w = (iz + 1)(z + i)^{-1}$, and $w = z$ will be denoted by S^2, S^3, and S^4, respectively. The last of these leaves every point in the plane unchanged. Such an operation we call an identical transformation. An identical transformation is usually assigned the symbol 1. Thus in the case just considered we write $S^4 = 1$.

48. The reader will find it an interesting and extremely valuable exercise to examine for himself other cases of linear transformation in detail. We may suggest for this purpose the transformations

$$w = \frac{z+1}{z-1}, \quad w = \frac{z-1}{z+1}, \quad w = \frac{z}{1-z}, \quad w = -\frac{1}{z},$$

all of which are of the elliptic type. $w = (z + 2)(2z + 1)^{-1}$ is a hyperbolic transformation with poles at ± 1.

49. In the preceding treatment of the examples of linear transformation we have considered the effect of the transformation on small curvilinear rectangles bounded by lines of motion and orthogonal curves. This treatment admits of an extension of great importance for the theory of mathematical physics. We may suppose *any* congruent systems of curves and the corresponding orthogonal system to be drawn in the plane, dividing it, as before, into small curvilinear rectangles, and consider the effect of any linear transformation on these rectangles.

A thorough examination of this part of the subject would detain us too long, but the student cannot be too strongly recommended to examine the matter, and thus more thoroughly to familiarize himself with the nature of the linear transformation. We can only suggest here a few cases which admit

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* See Plate III, Fig. 10.

of simple treatment. Thus, we may take a series of concentric circles about the point $+1$, and let the plane undergo any of the transformations mentioned above; for instance, the transformation $w = z^{-1}$. Another interesting configuration would be deduced by considering a system of hyperbolas and ellipses with their foci at the points ± 1, and letting the plane undergo the transformation $w = z^{-1}$, etc.*

5. *The conjugate transformations.*

50. In the treatment of the linear transformations of a complex variable we have employed as auxiliary operations the inversions and reflections of the complex plane. These are not themselves linear transformations, but only become such in combination. It is in several respects interesting to inquire whether these operations can be simply connected with the analytical treatment of a single complex variable, and we shall devote a brief space to the consideration of this question.

The operation of reflection with respect to the real axis is defined by the equation $w = \bar{z}$, or $\bar{w} = z$, where $\bar{z}$ means the complex number conjugate to z. Given now any linear transformation $w = (\alpha z + \beta)(\gamma z + \delta)^{-1}$, if we suppose this to be followed by a reflection on the real axis, the resultant operation is defined by the equation $\bar{w} = (\alpha z + \beta)(\gamma z + \delta)^{-1}$. We can obtain in this way a new system of transformations, also, ∞^6 in number, which we shall call conjugate transformations. We maintain that among these ∞^6 operations there are contained all inversions and all reflections of the plane.

For any inversion of the plane is reducible, as we have seen, to a succession of a translation, a contraction, an inversion with respect to the unit circle about the origin, an expansion, and a second translation. If in this series of operations we add to the inversion with respect to the unit circle a reflection on the real axis, the effect on the original transformation is to add to this a reflection on a straight line passing through the centre of the given inversion and parallel to the real axis. This operation is therefore reducible to a series of linear transformations, and is therefore itself a linear transformation. Now, if to this linear transformation we add again the reflection on the same line parallel to the real axis, the result is obviously that we return to the original inversion, which is therefore equivalent to a linear transformation followed by a reflection.

It will therefore be sufficient to show that all reflections of the plane on straight lines are contained among the operations $\bar{w} = (\alpha z + \beta)(\gamma z + \delta)^{-1}$.

* See Holzmüller. Theorie der Isogonalen Verwandtschaften.

For this purpose we make use of the fundamental property of reflection, that if a plane be reflected successively as two straight lines, the resultant operation is not a reflection, but a rotation of the plane about the point in which the two lines intersect. Thus* let AB and CB be the two fixed lines meeting in B at an angle φ, and let DB be any other line through B, making an angle ψ with AB and $\varphi - \psi$ with CB. Then by a reflection on CB this line DB is converted into the line $D'B$, which makes an angle $\varphi - \psi$ on the opposite side of CB, and consequently an angle $2\varphi - \psi$ with AB. If this line be then reflected on AB, it becomes the line $D''B$, which makes an angle $2\varphi - \psi$ with AB, and consequently an angle $(2\varphi - \psi) + \psi$ or 2φ with its initial position DB. Since this angle is independent of ψ, it appears that every line DB is turned about B through the same angle; i. e. that the entire plane is rotated about B through the angle 2φ.

In particular, if the two fixed lines are perpendicular to each other, the resultant of the two corresponding reflections is a rotation through 180°.

Suppose now that the line AB is the real axis while CB is any other line in the plane. Then a reflection on CB followed by a reflection on AB gives as a result a rotation of the plane about the point B. If now we add to this result a second reflection on AB, we return to the reflection on CB, which is therefore equivalent to a rotation about B followed by a reflection on AB.

Every reflection of the plane, and consequently every inversion, is therefore reducible to a linear transformation of the complex variable followed by a reflection on the real axis, and every such transformation is therefore defined by an equation of the form $\overline{w} = (\alpha z + \beta)(\gamma z + \delta)^{-1}$.

These are, however, not the only operations of this form, as is readily seen if we count the constants involved. Thus an inversion is determined by 3 constants, the co-ordinates of the centre and the radius of the fixed circle, and a reflection is determined by the 2 co-ordinates of the fixed line. But there are ∞^6 of the operations $\overline{w} = (\alpha z + \beta)(\gamma z + \delta)^{-1}$.

The general theory of these conjugate transformations is much more complicated than that of the linear transformation, and we shall only mention briefly the more prominent features. The complication in the theory arises partly from the fact, that the conjugate transformations have no lines of motion.† Again, the treatment of the linear transformations was simplified

* See Plate IV, Fig. 11.

† Thus, in the case of inversion, it is clear that no line can be drawn from a given point to the corresponding inverse point, such that all the points on this line shall be moved along the line, retaining their order of succession. For the point where the line cuts the circle of inversion remains fixed.

by the consideration of the two fixed poles of the motion. Now, if we seek for the fixed elements of a conjugate transformation, we must put, as before, $w = z$. This gives us

$$\bar{z} = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \text{or} \quad \gamma z\bar{z} + \delta\bar{z} - \alpha z - \beta = 0,$$

or

$$\gamma(x^2 + y^2) + \delta(x - yi) - \alpha(x + yi) - \beta = 0.$$

Writing, now, $\alpha = a_1 + a_2 i$, $\beta = b_1 + b_2 i$, $\gamma = c_1 + c_2 i$, $\delta = d_1 + d_2 i$, and separating the real and the imaginary parts, we have the two equations

$$c_1(x^2 + y^2) + d_1 x + d_2 y - a_1 x + a_2 y - b_1 = 0$$

and

$$c_2(x^2 + y^2) + d_2 x - d_1 y - a_2 x - a_1 y - b_2 = 0$$

to determine the fixed points of the transformation. These are the equations of two circles. These two circles may not be real, and, if real, they do not necessarily intersect, and, consequently, the given transformation may have no fixed points. If the circles intersect at all, they must intersect in two points, which will be the fixed points of the transformation. But the two circles may coincide, or the equation of one of them may disappear identically. The corresponding transformation then leaves the entire remaining circle fixed. Among these last transformations the inversion and reflection are obviously included.

The conjugate transformations seem to have been very little studied from this point of view.

One fundamental property of these transformations remains to be noted. The linear transformations have the property, that, if two of them be applied successively to the plane, the result is itself a linear transformation. Thus, if

$$z' = \frac{\alpha z + \beta}{\gamma z + \delta} \quad \text{and} \quad z'' = \frac{\alpha z' + \beta}{\gamma z' + \delta},$$

we have

$$z'' = \frac{\alpha' \dfrac{\alpha z + \beta}{\gamma z + \delta} + \beta'}{\gamma' \dfrac{\alpha z + \beta}{\gamma z + \delta} + \delta'} = \frac{(\alpha\alpha' + \beta'\gamma) z + \alpha'\beta + \beta'\delta}{(\alpha\gamma' + \gamma\delta') z + \beta\gamma' + \delta\delta'}.$$

The combination of two conjugate transformations, on the contrary, is not a conjugate transformation, but a linear transformation. Thus, if

$$\bar{z}' = \frac{\alpha z + \beta}{\gamma z + \delta} \quad \text{and} \quad \bar{z}'' = \frac{\alpha' z' + \beta'}{\gamma' z' + \delta'},$$

we may write the latter operation also in the form

$$z'' = \frac{\bar{\alpha}'\bar{z}' + \bar{\beta}'}{\bar{\gamma}'\bar{z}' + \bar{\delta}'}.$$

The combination of the two operations gives us, then,

$$z'' = \frac{\bar{\alpha}' \dfrac{\alpha z + \beta}{\gamma z + \delta} + \bar{\beta}'}{\bar{\gamma}' \dfrac{\alpha z + \beta}{\gamma z + \delta} + \bar{\delta}'} = \frac{(\bar{\alpha}'\alpha + \bar{\beta}'\gamma)\, z + \beta\bar{\alpha}' + \bar{\beta}'\delta}{(\alpha\bar{\gamma}' + \gamma\bar{\delta}')\, z + \beta\bar{\gamma}' + \delta\bar{\delta}'}.$$

This is a characteristic distinction between the two classes of operations.

The reader will easily show that any combination of a conjugate transformation with a linear transformation is a conjugate transformation. From this it will be clear that, instead of constructing the conjugate transformation from the linear transformation by adding to the latter the reflection on the real axis, we might have employed, in the place of this reflection, any other conjugate transformation.

6. *The representation of the complex quantities by means of points on a spherical surface.*

51. Thus far we have represented the complex quantities geometrically by means of points in a plane. The complex numbers constitute a two-dimensional system, and accordingly require for their geometrical representation a two-dimensional configuration. The plane is the simplest configuration of this species, and accordingly naturally presents itself first. But any other surface may be employed in the place of the plane for the same purpose. Among the various surfaces the simplest, after the plane, is the sphere, and, as the representation of the complex quantities on the spherical surface is closely connected with several important results of the modern mathematics, we shall next consider this representation in some detail.

In order that such a representation may be of utility for mathematical purposes, it is essential that the continuity of the system of complex num-

bers should be reproduced in the continuity of the geometrical representation. Thus, in the case of the plane, every set of values of the complex number which forms a continuous system is represented in the plane by points which form a continuous region. The same result is readily produced, in the case of the spherical surface, by the aid of the theory of stereographic projection

52. We begin by supposing the complex number to be represented in the plane in the usual way. The plane we will suppose to be held horizontal. On the plane we place the sphere so that it is tangent to the plane at the origin The point of tangency and the diametrically opposite point of the sphere we will call its south and north poles respectively.

If, now, we join each point of the plane with the north pole of the sphere by a straight line, this line will cut the sphere in one other point. This point we will call the *corresponding* point to the given point in the plane. The representation of the complex number on the spherical surface is now easily secured. If the complex number corresponding to any point in the plane be z, we have only to assign this number to the corresponding point on the sphere.

By this mode of representation to every complex number there corre sponds one point on the spherical surface, and, conversely, to every point on the spherical surface there corresponds one complex number.*

That the distribution of the complex numbers on the sphere reproduces the continuity of the analytical system is evident. The reader will form at very convenient conception of the distribution, if he imagines every point in the plane to be stamped with the value of the corresponding complex number and the same marks to be then transferred to the corresponding points on the spherical surface.

53. We proceed now to examine the character of the representation on the spherical surface more closely by the aid of our previous knowledge of the distribution in the complex plane. For this purpose we draw in the plane various configurations, mainly straight lines and circles, and determine the corresponding configurations on the sphere.

The bundle of rays through the origin in the plane evidently become meridian circles on the sphere, while the system of circles about the origin at a centre is converted into parallels of latitude. One of the circles, therefore becomes the equator of the sphere, and by properly choosing the radius on the sphere we can arrange that this circle shall be the unit circle. Every circle of the system which lies within the unit circle will then become a par-

* One apparent exception, presently to be noted, occurs in the case of the north pole of the sphere and the points at an infinite distance in the plane.

allel of latitude in the southern hemisphere, while those which lie outside the unit circle will become parallels in the northern hemisphere. As the radius of the circle in the plane is increased, the corresponding parallel will lie closer and closer to the north pole of the sphere. As the limiting case all the points at an infinite distance in the plane are converted into the north pole itself.

54. The point-to-point relation between the spherical surface and the plane fails, therefore, in this one case. But the correspondence between the system of complex numbers and the points in the spherical surface is not on this account in any way defective ; on the contrary, it is more complete ; for we have already seen that it is necessary for the purpose of the theory of functions to regard all the infinitely distant points of the plane as a single point. This requirement is geometrically realized in the case of the sphere, and no convention on this point is longer necessary. On account of the complete agreement of the geometrical and analytical theories, the representation on the spherical surface is in several respects simpler and more convenient than that in the plane.

55. Returning to the correspondence of figures in the plane and on the sphere, we consider next the case of any straight line and of any circle in the plane. The lines joining the points of a straight line to the north pole of the sphere lie in a plane which cuts the sphere in a circle passing through the north pole. Every straight line in the complex plane is therefore converted into such a circle on the sphere. Again, the lines joining the points of a circle to the north pole of the sphere lie on the surface of an oblique circular cone. By the proposition of Apollonius every such cone has two sets of circular sections, which are called subcontrary to each other. We assert that the curves of intersection of the cone with the sphere is one of the circular sections of the cone which are subcontrary to the given circle in the complex plane.

The proof follows at once from the fundamental property of subcontrary sections. Thus in the figure* let ABC be any circle in the complex plane, A and B being the points of the circle nearest to and farthest from the origin. Let P be the north pole of the sphere, and $PSRO$ be the section of the sphere cut out by the plane POB. The figure will otherwise explain itself. Now RS will be the diameter of a circular section of the cone subcontrary to ABC, if $\alpha' = \beta$ and, consequently, $\beta' = \alpha$.

But

$$\beta = \tfrac{1}{2} \text{ arc } OP - \tfrac{1}{2} \text{ arc } OS = 90^\circ - \tfrac{1}{2} \text{ arc } OS,$$

* See Plate IV, Fig. 13.

and

$$\alpha' = \tfrac{1}{2} \text{ arc } SP = \tfrac{1}{2} \text{ arc } OP - \tfrac{1}{2} \text{ arc } OS = \beta.$$

RS and AB are then diameters of subcontrary circular sections of the cone. But the plane of the circular section in which RS lies will also cut the sphere in a circle, of which RS will also be a diameter. This plane therefore cuts the cone and the sphere in the same circle, which is therefore the intersection of the cone and the sphere.

56. Every straight line and every circle in the plane becomes, therefore, a circle on the spherical surface, which in the case of the straight line is only distinguished by the fact that it passes through the north pole of the sphere. Geometrically the north pole is not different from any other point on the spherical surface, and there is no essential distinction between these two classes of circles. The convention which it was necessary to adopt in the case of the complex plane, that no distinction in kind should be made between straight lines and circles, but that the straight lines should be regarded merely as a special class of circles which pass through the point at infinity, therefore falls away as superfluous in the case of the sphere, as indeed do all conventions which were needed to adapt the geometry of the plane to the system of complex quantities.

57. A further important property of stereographic projections is likewise readily obtainable by purely geometrical considerations. Thus,* if A be any point in the complex plane, and S the corresponding point on the sphere, and if ST be the trace of the tangent plane at S on the plane POA, we have, in the figure,

$$\beta = \tfrac{1}{2} \text{ arc } SP, \quad \alpha = \tfrac{1}{2} \text{ arc } OP - \tfrac{1}{2} \text{ arc } OS = \tfrac{1}{2} \text{ arc } SP.$$

Hence $\alpha = \beta$. The tangent plane at S and the complex plane therefore make the same angle with the line of projection PA. If, now, we draw any two straight lines in the complex plane through A, these will become circles through S on the sphere. The tangents to these circles will be the intersections of the tangent plane at S with the two planes of projection. But on account of the symmetry of the complex plane and the tangent plane with respect to the line PA, it is evident that the angle between the two tangents at S is equal to the angle between the given straight lines in the complex plane. It appears, therefore, that every angle in the plane is equal to the corresponding angle on the spherical surface; that is, the stereographic pro-

* See Plate IV, Fig. 14.

jection, like the linear transformation in the plane, is a conformal transformation.*

58. We establish, next, the analytical theory of the relation between the representation of the complex number in the plane and that on the sphere. For this purpose we introduce a Cartesian co-ordinate system in space. As the origin we take the origin in the plane, and, for axes of two of the co-ordinates ξ and η, the real and imaginary axes respectively in the plane. The third axis, that of ζ, is taken at right angles to the plane, and so that its positive direction is upward.

For convenience we assume the radius of the complex sphere to be $\frac{1}{2}$. The equation of the sphere is, then,

$$\xi^2 + \eta^2 + (\zeta - \tfrac{1}{2})^2 = \tfrac{1}{4}, \quad \text{or} \quad \xi^2 + \eta^2 + \zeta(\zeta - 1) = 0.$$

If, then, x and y be the Cartesian co-ordinates of any point in the plane, and ξ, η, and ζ those of the corresponding point on the sphere, we have, at once, from the theory of similar triangles,

$$\frac{\xi^2 + \eta^2}{1 - \zeta} = \frac{x^2 + y^2}{x^2 + y^2 + 1} \quad \text{and} \quad \frac{\xi}{\eta} = \frac{x}{y}.$$

Combining these equations with the equation of the sphere, we readily deduce

$$\zeta = \frac{x^2 + y^2}{x^2 + y^2 + 1}, \qquad \xi = \frac{x}{y}\eta;$$

$$\therefore \left[1 + \frac{x^2}{y^2}\right]\eta^2 + \frac{x^2 + y^2}{x^2 + y^2 + 1}\left[\frac{-1}{x^2 + y^2 + 1}\right] = 0;$$

$$\therefore \eta^2 = \left[\frac{y}{x^2 + y^2 + 1}\right]^2, \quad \eta = \frac{y}{x^2 + y^2 + 1}, \quad \xi = \frac{x}{x^2 + y^2 + 1}.$$

The three equations

$$\xi = \frac{x}{x^2 + y^2 + 1}, \quad \eta = \frac{y}{x^2 + y^2 + 1}, \quad \zeta = \frac{x^2 + y^2}{x^2 + y^2 + 1}$$

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* Strictly speaking, it is necessary to show also that the ratio of similarity is constant about any point. The reader can easily supply this omission.

give the values of ξ, η, ζ in terms of x and y. From these we have, also,

$$x^2 + y^2 = \frac{\zeta}{1 - \zeta}, \qquad \xi = \frac{x}{\dfrac{\zeta}{1-\zeta} + 1} = x(1 - \zeta);$$

$$\therefore x = \frac{\xi}{1 - \zeta}. \quad \text{Similarly, } y = \frac{\eta}{1 - \zeta}.$$

These equations,

$$x = \frac{\xi}{1 - \zeta}, \qquad y = \frac{\eta}{1 - \zeta}, \qquad x^2 + y^2 = \frac{\zeta}{1 - \zeta},$$

give the values of x and y in terms of ξ, η, ζ. Since on the equator of the sphere $\zeta = \frac{1}{2}$, we have for the circle in the plane corresponding to the equator $x^2 + y^2 = 1$. That is, this is the unit circle.

59. We can now readily show analytically that every straight line and every circle in the plane becomes a circle on the sphere. For if the equations of a straight line and of a circle be

$$ax + by + c = 0 \quad \text{and} \quad x^2 + y^2 + 2gx + 2fy + c = 0,$$

we have, from the preceding equations, for the corresponding curves on the spherical surface,

$$a\xi + b\eta + c(1 - \zeta) = 0 \quad \text{and} \quad \zeta + 2g\xi + 2f\eta + c(1 - \zeta) = 0.$$

Both these equations represent planes, which accordingly cut the sphere in circles.

60. We have now at hand all the geometrical and analytical apparatus necessary for the treatment of the interpretation on the spherical surface of the linear transformations of the complex variable.

We consider the transformations first from the purely geometrical standpoint. Their interpretation from this point of view is not always so simple as in the case of the plane, although important exceptions occur.

The cases of rotation and expansion about the origin in the complex plane require little description. In the case of the rotation the lines of motion and the orthogonal system are concentric circles about the origin and the bundle of rays through the origin respectively. To them correspond on the spherical surface the circles of latitude and the meridians. The corresponding transformation of the spherical surface is evidently a rotation about the vertical axis.

On the other hand, to an expansion about the origin will correspond a motion of points on the sphere along the meridians from the south toward the north pole. The circles of latitude will remain such, but will be moved northward.

61. If, now, we combine these two motions, so as to produce the logarithmic spiral motion in the plane, the corresponding motion on the sphere will also be a species of spiral motion. Points are carried away from the south pole toward the north pole, while at the same time they circle about the sphere. Since the logarithmic spiral in the plane cuts all rays through the origin at the same angle, and since the stereographic projection leaves every angle unchanged, it follows that the spiral on the sphere cuts all the meridians at the same angle. This spiral is, therefore, the Loxodromic Curve.

62. In the case of translation in the plane the lines of motion and the orthogonal systems are two systems of straight lines. The corresponding curves on the spherical surface are two orthogonal systems of circles which pass through the north pole of the sphere. The circles of each system are tangent to each other at the north pole, but the two common tangents are at right angles to each other. The transformation on the spherical surface will consist in a motion along one of the systems of circles. From the figure* the parabolic character of the motion is at once apparent.

63. If the rotation, expansion, or logarithmic spiral motion take place about any other point in the plane except the origin, the corresponding transformations on the sphere are somewhat more complicated. To the system of rays through the fixed point in the plane corresponds a system of circles on the sphere passing through the corresponding point and the north pole. The circles in the plane about the fixed point as a centre become a second system of circles on the sphere orthogonal to the first set. To a rotation in the plane will then correspond on the sphere a species of double circular motions about the fixed point and the north pole, while to the expansion in the plane will correspond a species of hyperbolic motions on the sphere, and, finally, to the logarithmic spiral motion in the plane will correspond a species of double spiral motions on the sphere. These motions on the sphere are, accordingly, less simple than those in the plane.

64. The operation of reciprocation, on the other hand, has a remarkably simple interpretation in the case of the spherical distribution. In the plane this operation leaves the points ± 1 fixed. Since the unit circle in the plane becomes the equator of the sphere, and since the points evidently become

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* See Plate IV, Fig. 14.

opposite points on the spherical surface, it appears that the transformation of the sphere leaves the extremities of the horizontal diameter parallel to the axis of ξ unchanged. Again, the circles in the plane through the points ± 1 become great circles of the sphere through the corresponding points, and, if we regard these points for the moment as poles of the sphere, the circles in the plane about the points ± 1 become circles on the sphere orthogonal to the preceding system; i. e. they become latitude circles of the sphere.

The operation of reciprocation will therefore consist in a rotation of the sphere about the diameter parallel to the axis of ξ. Moreover, as a repetition of the operation restores every point to its initial position, the rotation must be a rotation through 180°. The north and south poles of the sphere are accordingly interchanged, as is also evident from the fact that in the plane the points 0 and ∞ are interchanged.

The operation of reciprocation, therefore, leaves the spherical surface entirely unchanged in all its parts, and simply rotates it bodily through 180°.

65. The cases of the transformations of the spherical surface which correspond to the double spiral, elliptic, hyperbolic, and parabolic motions in the plane are now readily disposed of. The lines of motion for these transformations in the plane were obtained by reciprocating the various forms of the single spiral. Accordingly, the corresponding lines of motion on the sphere will be obtained from those which we have already considered by rotating the sphere about the diameter parallel to the ξ axis through 180°. These new systems of lines of motion will not, therefore, differ in any way from those already considered.

We have, therefore, already obtained all the forms of motion on the spherical surface which correspond to any linear transformation of the complex variable. Every such motion is either a double spiral motion, or an elliptic, hyperbolic, or parabolic motion. The distinction between the various forms of the double spiral motion, on the one hand, and those of the single spiral motion, on the other, has completely disappeared, this result being due to the fact that the "point at infinity" is on the spherical surface actually what it purports to be, a single geometrical point.

The distinction between these two classes of motions is based on the geometry of the complex plane, and is not in any way inherent in the nature of the algebraic system of complex numbers. For distributions of the complex numbers on other surfaces this distinction accordingly disappears, and is replaced by other classifications dependent in each case on the geometry of the given surface. Thus in the case of the spherical surface certain of the transformations are evidently of a simpler geometrical character than others.

In particular, those for which the two fixed points are diametrically opposite each other form an especially interesting system, the theory of which will be presently discussed. The motions of this system take place along loxodromic curves, and the system includes in particular all the *rotations* of the sphere.

66. The analytical expressions in terms of the space co-ordinates ξ, η, and ζ of the transformations of the spherical surface corresponding to a linear transformation of the complex variable are peculiarly simple in form; and as their theory leads to some of the most important developments of the modern Algebra and Geometry, we shall devote some considerable space here to their treatment.

If x and y, ξ, η, and ζ be the co-ordinates of two corresponding points, one in the plane the other on the sphere, and if x' and y', ξ', η', and ζ' be the co-ordinates of the same points transformed, we have always

$$x = \frac{\xi}{1-\zeta}, \quad y = \frac{\eta}{1-\zeta}, \quad x^2 + y^2 = \frac{\zeta}{1-\zeta};$$
$$x' = \frac{\xi'}{1-\zeta'}, \quad y' = \frac{\eta'}{1-\zeta'}, \quad x'^2 + y'^2 = \frac{\zeta'}{1-\zeta'},$$

where the pairs of values x', y' and x, y are connected by the linear transformation of the complex variable $z = x + yi$. We have to determine ξ', η', and ζ' in terms of ξ, η, and ζ.

We begin with the case of the general logarithmic spiral motion in the plane defined by the equation $z' = az + \beta$, where we will suppose $a = a_1 + a_2 i$, $\beta = b_1 + b_2 i$. We have, then,

$$x' + y'i = (a_1 + a_2 i)(x + yi) + b_1 + b_2 i.$$

Hence

$$x' = a_1 x - a_2 y + b_1,$$
$$y' = a_1 y + a_2 x + b_2,$$
$$x'^2 + y'^2 = (a_1^2 + a_2^2)(x^2 + y^2) + 2(a_1 b_1 + a_2 b_2)x + 2(a_1 b_2 - a_2 b_1)y + b_1^2 + b_2^2.$$

Hence from the equations connecting the x, y with the ξ, η, ζ, we have

$$\frac{\xi'}{1-\zeta'} = \frac{a_1\xi - a_2\eta + b_1(1-\zeta)}{1-\zeta}, \tag{1}$$

$$\frac{\eta'}{1-\zeta'} = \frac{a_2\xi + a_1\eta + b_2(1-\zeta)}{1-\zeta}, \tag{2}$$

$$\frac{\zeta'}{1-\zeta'} = \frac{(a_1^2 + a_2^2)\zeta + 2(a_1 b_1 + a_2 b_2)\xi + 2(a_1 b_2 - a_2 b_1)\eta + (b_1^2 + b_2^2)(1-\zeta)}{1-\zeta} \tag{3}$$
$$= \frac{(a_1^2 + a_2^2 - b_1^2 - b_2^2)\zeta + 2(a_1 b_1 + a_2 b_2)\xi + 2(a_1 b_2 - a_2 b_1)\eta + b_1^2 + b_2^2}{1-\zeta}.$$

If we denote the numerator on the right-hand side of the third equation by F, we have

$$\frac{\zeta'}{1-\zeta'}=\frac{F}{1-\zeta}; \quad \therefore \zeta'=\frac{F}{1-\zeta}(1-\zeta'),$$

or

$$\zeta'=\frac{F}{1-\zeta+F}, \quad 1-\zeta'=\frac{1-\zeta}{1-\zeta+F}.$$

Also

$$\xi'=\frac{a_1\xi-a_2\eta+b_1(1-\zeta)}{1-\zeta}(1-\zeta');$$

$$\therefore \xi'=\frac{a_1\xi-a_2\eta+b_1(1-\zeta)}{1-\zeta+F},$$

and

$$\eta'=\frac{a_2\xi+a_1\eta+b_2(1-\zeta)}{1-\zeta+F}.$$

The co-ordinates ξ', η', ζ' are therefore *linear* functions of ξ, η, ζ, as indeed is at once apparent from equations (1), (2), and (3). But the fact is also to be especially noticed, on account of its importance later, that the denominators in the expression for ξ', η', ζ' are all equal.

The transformation of the spherical surface corresponding to the transformation $z'=\alpha z+\beta$ is therefore analytically, when expressed in terms of the co-ordinates ξ, η, ζ, itself a *linear* transformation of these co-ordinates, and herein lies its simplicity.

Again, in the case of reciprocation, we have

$$z'=\frac{1}{z}, \quad \text{or} \quad x'+y'i=\frac{x-yi}{x^2+y^2};$$

$$\therefore x'=\frac{x}{x^2+y^2}, \quad y'=\frac{-y}{x^2+y^2}, \quad x'^2+y'^2=\frac{1}{x^2+y^2}.$$

$$\frac{\xi'}{1-\zeta'}=\frac{\xi}{\zeta}, \tag{1}$$

$$\frac{\eta'}{1-\zeta'}=\frac{\eta}{\zeta}, \tag{2}$$

$$\frac{\zeta'}{1-\zeta'}=\frac{1-\zeta}{\zeta}. \tag{3}$$

From (3) we have

$$\zeta'=1-\zeta; \quad \therefore \xi'=\xi, \quad \eta'=\eta.$$

These, again, are linear equations. The character of the transformation as a rotation of the sphere is analytically at once apparent.

67. All the other linear transformations of a complex variable were obtained by a combination of those which we have just considered. Accordingly, all the corresponding transformations of the spherical surface will be obtained by combining those which we have now obtained. But it is evident that a combination of any two linear transformations of the ξ, η, ζ will itself be a linear transformation. And, moreover, the denominators of the fractions which occur on the right-hand sides of the equations of transformations will all be equal. We have, therefore, this fundamental proposition: *Every transformation of the spherical surface which arises from a linear transformation of the complex variable is, when analytically expressed in terms of the co-ordinates ξ, η, ζ, itself a linear transformation of these co-ordinates, in which the denominators in the equations of transformation are all equal.*

A very slight extension of the preceding theory will now enable us to make an interesting connection with a portion of the projective geometry of three-dimensional space. We have thus far considered the transformation of the co-ordinates ξ, η, ζ as affecting only the spherical surface. But ξ, η, ζ are space co-ordinates, and any operation performed on them indicates a transformation not only of the spherical surface, but also of the entire space in which the complex sphere is situated. The geometrical character of these space transformations is readily ascertained. They are linear transformations of the ξ, η, ζ with common denominators. The most general transformation of this character may be written

$$\begin{aligned}\xi' &= \frac{a_1\xi + b_1\eta + c_1\zeta + d_1}{a_4\xi + b_4\eta + c_4\zeta + d_4},\\ \eta' &= \frac{a_2\xi + b_2\eta + c_2\zeta + d_2}{a_4\xi + b_4\eta + c_4\zeta + d_4},\\ \zeta' &= \frac{a_3\xi + b_3\eta + c_3\zeta + d_3}{a_4\xi + b_4\eta + c_4\zeta + d_4}.\end{aligned} \tag{1}$$

If we solve these equations for ξ, η, ζ we obtain the reverse transformations, which will also be linear with a common denominator.

$$\begin{aligned}\xi &= \frac{a_1'\xi' + b_1'\eta' + c_1'\zeta' + d_1'}{a_4'\xi' + b_4'\eta' + c_4'\zeta' + d_4'},\\ \eta &= \frac{a_2'\xi' + b_2'\eta' + c_2'\zeta' + d_2'}{a_4'\xi' + b_4'\eta' + c_4'\zeta' + d_4'},\\ \zeta &= \frac{a_3'\xi' + b_3'\eta' + c_3'\zeta' + d_3'}{a_4'\xi' + b_4'\eta' + c_4'\zeta' + d_4'};\end{aligned} \tag{2}$$

where the a', b', c', d' are rational fractions of the third degree in the a, b, c, d.

68. To determine the geometrical effect of these transformations on the three-dimensional space, we consider the case of any plane $A\xi + B\eta + C\zeta + D = 0$. If in this equation we substitute for ξ, η, ζ the expression given in (2), we have a new equation of the form, $A'\xi' + B'\eta' + C'\zeta' + D' = 0$, where A', B', C', D' are linear functions of A, B, C, D and the a', b', c', d'. This new equation is, again, the equation of a plane. It appears, therefore, that our transformation converts every plane in space into a plane. Two planes have their line of intersection in common. This will be converted into the common element of the two transformed planes; i. e. into their line of intersection. Every straight line in space is therefore converted into a straight line. Such transformations are called *collineations*.

69. All our transformations are then collineations of space. Conversely, every collineation of space is defined by equation of the form (1) or (2). These equations contain 16 arbitrary constants. We may, however, without loss of generality, assume that any one of these, say a'_4, $= 1$. There remain 15 constants, any one of which may take any value from $-\infty$ to $+\infty$. In other words, there are ∞^{15} real collineations of space.

70. Among these ∞^{15} collineations are contained all the ∞^6 transformations of space which arise from the linear transformation of the complex variable. The criterion by which these ∞^6 transformations are distinguished among the ∞^{15} is geometrically obvious. The ∞^6 transformations are not only collineations of space, but they have also the property, that they convert the spherical surface into itself. The general equation of a surface of the second degree in space of three dimensions is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

and contains therefore 10 constants. We may, however, assume $A = 1$, when there remain 9 essential constants. Every linear transformation obviously converts a surface of any order into another surface of the same order. If, now, it be required, in particular, that a surface of the second order shall be converted into itself, the 9 coefficients of the equations of the given surface and the 9 coefficients of the equations of the transformed surface must be equal each to each. This requires, then, 9 equations of condition among the ∞^{15} constants of the general linear transformation. There remain, therefore, 6 independent constants; i. e. there are ∞^6 collineations of space which transform a given surface of a second order, in particular a sphere, into itself.

71. There were also ∞^6 collineations of space which arose from the ∞^6 linear transformation of the complex variable, and which also convert the

spherical surface into itself. It does not, however, follow that these two systems are identical, operation for operation; but the second system must be contained in the first, which may, however, include other collineations. That there are collineations of space which convert the sphere into itself, but which do not correspond to linear transformations of the complex variable, is clear; for a reflection of space on any diametral plane of the sphere, for instance on the plane of $\xi\zeta$, evidently converts the spherical surface into itself. Such a reflection is a collineation; in fact, for the case when the fixed plane is that of $\xi\zeta$, it is defined by the equations $\xi' = \xi$, $\eta' = -\eta$, $\zeta' = \zeta$. But if we return by the aid of stereographic projection to the complex plane, this reflection is replaced by a reflection on the real axis of the plane, and this is not a linear transformation of the complex variable. Again, if we combine this reflection of the sphere on the plane of $\xi\zeta$ with each of the ∞^6 collineations which arise from the linear transformations, these combinations will also be collineations, and will convert the sphere into itself. These compound operations, interpreted in terms of the complex variable, are the conjugate transformations.

72. It appears, therefore, that the collineations of space which convert a sphere into itself are at least twice as numerous as those which arise from the linear transformations of the complex variable. A full investigation of the question, whether there are still further collineations in the system, would be too extensive for insertion here. In the case of a particular class of these collineations we shall treat this matter in detail in a later section. For the present, we shall content ourselves with the following assertions, which will serve also to indicate the direction which the general investigation follows:

In considering the collineations of space which convert a given sphere into itself, we will suppose the origin to be taken at the centre of the sphere. *If this be the case, the square of the determinant*

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

is equal to $+1$, *and accordingly the determinant itself is equal to either* $+1$ *or* -1.

All these collineations may therefore be divided into two equal systems. Those collineations whose determinant is $+1$ *are identical in their effect on the spherical surface with the linear transformation of a complex variable distrib-*

uted on the surface. Those collineations whose determinant is -1 *are identical with the conjugate transformations of the same complex variable.*

73. If the fixed points of a transformation of the spherical surface corresponding to a linear transformation of the complex variable be given, the equation for the latter transformation is at once obtainable by the aid of the formula of Sec. 36,

$$\frac{w-\gamma}{w-\delta} = \varepsilon \frac{z-\gamma}{z-\delta}.$$

Thus, if ξ, η, ζ and ξ', η', ζ' be the co-ordinates of the fixed points on the sphere, and if $\gamma = x + yi$ and $\delta = x' + y'i$, we have

$$\gamma = \frac{\xi + \eta i}{1-\zeta}, \quad \delta = \frac{\xi' - \eta' i}{1-\zeta'}.$$

Substituting these values of γ and δ in the formula above, we have for the required equation

$$\frac{w(1-\zeta) - (\xi + \eta i)}{w(1-\zeta') + (\xi' + \eta' i)} = \varepsilon \frac{z(1-\zeta) - (\xi + \eta i)}{z(1-\zeta') - (\xi' + \eta' i)}.$$

74. Among the ∞^6 collineations which arise from linear transformations of the complex variable, those are of especially simple character for which the two fixed points are diametrically opposite. Their number is readily ascertained. Thus, one of the fixed points may lie anywhere on the spherical surface, and since a surface contains ∞^2 points, we may select this one pole in ∞^2 different ways. The other pole is then fixed also. There remains the constant ε, which again involves two arbitrary constants. There are, therefore, ∞^4 collineations of this kind.

Among these, those for which the modulus of ε is 1 are *rotations* of the sphere, and of these there are accordingly ∞^3.

75. The corresponding transformations of the complex variable are obtained from the formula of Art. 70, by putting $\xi' = -\xi$, $\eta' = -\eta$, $\zeta' = 1 - \zeta$, this being the condition that the two points ξ, η, ζ and ξ', η', ζ' shall be diametrically opposite. The resulting equation is

$$\frac{w(1-\zeta) - (\xi + \eta i)}{w\zeta + (\xi + \eta i)} = \varepsilon \frac{z(1-\zeta) - (\xi + \eta i)}{z\zeta + (\xi + \eta i)},$$

which is a loxodromic motion or a rotation, according as modulus ε is not or is equal to 1.

76. The relation between the complex numbers which correspond to diametrically opposite points of the spherical surface is readily deduced. Thus, if $x + yi$ and $x' + y'i$ be the complex numbers corresponding to two such points, and if ξ, η, ζ and $-\xi$, $-\eta$, $1-\zeta$ be their space co-ordinates, we have

$$x = \frac{\xi}{1-\zeta}, \quad y = \frac{\eta}{1-\zeta}, \quad x^2 + y^2 = \frac{\zeta}{1-\zeta},$$

$$x' = -\frac{\xi}{\zeta}, \quad y' = -\frac{\eta}{\zeta}, \quad x'^2 + y'^2 = \frac{1-\zeta}{\zeta};$$

$$\therefore x'^2 + y'^2 = \frac{1}{x^2+y^2}, \quad \frac{x'}{x} = -\frac{1}{x^2+y^2}, \quad \frac{y'}{y} = -\frac{1}{x^2+y^2};$$

$$\therefore x' + y'i = -\frac{x+yi}{x^2+y^2}.$$

That is, the corresponding points in the complex plane lie on a straight line through the origin, on opposite sides of the origin, and at distances from the origin whose product is 1.

77. On account of the importance of the theory of the rotations of the sphere for many branches of modern mathematics, and because we shall have to employ these rotations in following articles, we treat here some of the more important portions of this theory.

For the sake of symmetry, it will be convenient to assume the origin to be at the centre of the sphere.* Every rotation of space converts any finite point into a finite point. Accordingly, in the equation of Art. 64 the denominators must reduce to their constant terms, for otherwise all points in the plane $a_4\xi + b_4\eta + c_4\zeta + d_4 = 0$ would give infinite values of ξ', η', ζ'; that is, these points would become points at infinity. We may also assume that $d_4 = 1$. The equations for a rotation, therefore, reduce to the form

$$\xi' = a_1\xi + b_1\eta + c_1\zeta + d_1,$$
$$\eta' = a_2\xi + b_2\eta + c_2\zeta + d_2,$$
$$\zeta' = a_3\xi + b_3\eta + c_3\zeta + d_3.$$

That is, the transformations are in this case integral in terms of the space

* One result of this change of origin will be that the formulæ of Art. 55 will become

$$\xi = \frac{x}{x^2+y^2+1}, \quad \eta = \frac{y}{x^2+y^2+1}, \quad \zeta = \frac{x^2+y^2-1}{2(x^2+y^2+1)},$$

as the reader can easily verify.

co-ordinates. Since, moreover, the centre of the sphere for which $\xi=\eta=\zeta=0$ remains fixed, we must have also $d_1=d_2=d_3=0$.

Again, the equation of the sphere is $\xi^2+\eta^2+\zeta^2=\frac{1}{4}$, and this must be transformed into itself. This requires

$$\begin{aligned} a_1^2+a_2^2+a_3^2&=1, & a_1b_1+a_2b_2+a_3b_3&=0,\\ b_1^2+b_2^2+b_3^2&=1, & b_1c_1+b_2c_2+b_3c_3&=0,\\ c_1^2+c_2^2+c_3^2&=1, & a_1c_1+a_2c_2+a_3c_3&=0. \end{aligned}$$

The equations of transformation are, therefore, the same as those for orthogonal change of co-ordinates, as was to be expected. These equations contain 9 coefficients, and these are connected by 6 relations. There are, therefore, 3 independent constants, and consequently ∞^3 transformations, in the system, as we have already seen must be the case.

Among these are included ∞^3 rotations. But all reflections of space on the diametral planes of the sphere, and consequently all combinations of these reflections with the rotations, are also included in the system. For these reflections also convert every finite point into a finite point and leave the centre of the sphere fixed.

78. A simple criterion serves to divide the integral collineations into two systems.

The determinant

$$\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3 \end{vmatrix}$$

is called the determinant of the transformation. The square of this determinant is the determinant

$$\begin{vmatrix} a_1^2+a_2^2+a_3^2 & a_1b_1+a_2b_2+a_3b_3 & a_1c_1+a_2c_2+a_3c_3\\ a_1b_1+a_2b_2+a_3b_3 & b_1^2+b_2^2+b_3^2 & b_1c_1+b_2c_2+b_3c_3\\ a_1c_1+a_2c_2+a_3c_3 & b_1c_1+b_2c_2+b_3c_3 & c_1^2+c_2^2+c_3^2 \end{vmatrix}.$$

In this determinant all the terms in the principal diagonal are equal to 1, while all the other terms are 0. Consequently, the determinant is equal to 1, and therefore the determinant of the transformation is either $+1$ or -1.*

* The following important relations among the coefficients may also be deduced: If the determinant is $+1$,

$$\begin{aligned} a_1&=b_2c_3-b_3c_2, & b_1&=a_2c_3-a_3c_2, & c_1&=a_2b_3-a_3b_2,\\ a_2&=b_3c_1-b_1c_3, & b_2&=a_1c_3-a_3c_1, & c_2&=a_3b_1-a_1b_3,\\ a_3&=b_1c_2-b_2c_1; & b_3&=a_2c_1-a_1c_2; & c_3&=a_1b_2-a_2b_1; \end{aligned}$$

and if the determinant is -1 the same equations hold, except that the signs of the right members are all to be changed.

It is a known property of integral linear transformations that, if two such transformations occur successively, the determinant of their resultant is the product of the determinants of the two components. Accordingly, the resultant of two of the above transformations of determinant $+1$ has itself the determinant $+1$; and, if the two transformations have the determinant -1, their resultant has still the determinant $+1$.

We divide our transformations into two systems, according as their determinant is $+1$ or -1. The resultant of two transformations of the first system is, then, itself a transformation of the first system, and the resultant of two transformations of the second system is a transformation of the first system. The combination of any operation of the first system with any operation of the second system leads, on the other hand, to an operation of the second system.

In particular, we may notice that every operation of the second system is equivalent to an operation of the first system followed by a reflection on any diametral plane, say that of $\eta\zeta$. For the latter transformation amounts to changing the sign of a_1, a_2, and a_3, and consequently of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

79. The analogy to the theory of the linear and the conjugate transformations is now sufficiently obvious. We maintain, in agreement with the assertions of Art. 69 :—

All transformations of the first system (determinant $+1$) are rotations of space about the origin, and are equivalent in their effect on the surface of the sphere to linear transformations of a complex variable whose poles satisfy the conditions of Art. 73, and for which the modulus of ε is 1.

All transformations of the second system (determinant -1) are combinations of the rotations with a reflection on any diametral plane, and are equivalent in their effect on the surface of the sphere to the transformations conjugate to the linear transformations above.

It will be sufficient to demonstrate the former proposition; the latter then follows at once. The equations connecting the co-ordinates in the plane with those in space are

$$\xi = \frac{x}{x^2+y^2+1}, \quad \eta = \frac{y}{x^2+y^2+1}, \quad \zeta = \frac{x^2+\eta^2-1}{2(x^2+y^2+1)}.*$$

* See foot-note, page 169.

Introducing these and the corresponding accented formulæ in the equations

$$\xi' = a_1\xi + b_1\eta + c_1\zeta,$$
$$\eta' = a_2\xi + b_2\eta + c_2\zeta,$$
$$\zeta' = a_3\xi + b_3\eta + c_3\zeta,$$

of Art. 74, we have

$$\frac{x'}{x'^2 + y'^2 + 1} = \frac{2a_1x + 2b_1y + c_1(x^2 + y^2 - 1)}{2(x^2 + y^2 + 1)}, \tag{1}$$

$$\frac{y'}{x'^2 + y'^2 + 1} = \frac{2a_2x + 2b_2y + c_2(x^2 + y^2 - 1)}{2(x^2 + y^2 + 1)}, \tag{2}$$

$$\frac{x'^2 + y'^2 - 1}{2(x'^2 + y'^2 + 1)} = \frac{2a_3x + 2b_3y + c_3(x^2 + y^2 - 1)}{2(x^2 + y^2 + 1)}. \tag{3}$$

From the last equation we obtain

$$\frac{1}{x'^2 + y'^2 + 1} = \frac{-2a_3x - 2b_3y + (1 - c_3)(x^2 + y^2) + 1 + c_3}{2(x^2 + y^2 + 1)}; \tag{4}$$

and dividing (1) and (2) by (4),

$$x' = \frac{2a_1x + 2b_1y + c_1(x^2 + y^2) - c_1}{-2a_3x - 2b_3y + (1 - c_3)(x^2 + y^2) + 1 + c_3}, \tag{5}$$

$$y' = \frac{2a_2x + 2b_2y + c_2(x^2 + y^2) - c_2}{-2a_3x - 2b_3y + (1 - c_3)(x^2 + y^2) + 1 + c_3}. \tag{6}$$

We have asserted that these two equations are equivalent to a linear transformation of the complex variable $z = x + yi$; i. e. that they can be reduced to the form $z' = (\alpha z + \beta)(\gamma z + \delta)^{-1}$.

In this last equation we will suppose, for simplicity, that $\delta = 1$. We will also let

$$\alpha = \alpha_1 + \alpha_2 i, \quad \beta = \beta_1 + \beta_2 i, \quad \gamma = \gamma_1 + \gamma_2 i,$$

where the letters with subscripts are real. The equation then may be written

$$x' + y'i = \frac{(\alpha_1 + \alpha_2 i)(x + yi) + \beta_1 + \beta_2 i}{(\gamma_1 + \gamma_2 i)(x + yi) + 1} = \frac{\alpha_1 x - \alpha_2 y + \beta_1 + i(\alpha_1 y + \alpha_2 x + \beta_2)}{\gamma_1 x - \gamma_2 y + 1 + i(\gamma_1 y + \gamma_2 x)}$$
$$= \frac{[\alpha_1 x - \alpha_2 y + \beta_1 + i(\alpha_1 y + \alpha_2 x + \beta_2)][\gamma_1 x - \gamma_2 y + 1 - i(\gamma_1 y + \gamma_2 x)]}{(\gamma_1 x + \gamma_2 y + 1)^2 + (\gamma_1 y + \gamma_2 x)^2}.$$

From which, separating the real and the imaginary parts, we have

$$x' = \frac{(\alpha_1\gamma_1 + \alpha_2\gamma_2)(x^2+y^2) + (\alpha_1 + \beta_1\gamma_1 + \beta_2\gamma_2)x - (\alpha_2 + \beta_1\gamma_2 - \beta_2\gamma_1)y + \beta_1}{(\gamma_1^2+\gamma_2^2)(x^2+y^2) + 2\gamma_1 x - 2\gamma_2 y + 1}, \quad (5')$$

$$y' = \frac{(\alpha_2\gamma_1 - \alpha_1\gamma_2)(x^2+y^2) + (\alpha_2 + \beta_2\gamma_1 - \beta_1\gamma_2)x + (\alpha_1 - \beta_1\gamma_1 - \beta_2\gamma_2)y + \beta_2}{(\gamma_1^2+\gamma_2^2)(x^2+y^2) + 2\gamma_1 x - 2\gamma_2 y + 1}. \quad (6')$$

The equations (5′) and (6′) must be identical with (5) and (6). If we consider the denominators, we must, then, have

$$\frac{1-c_3}{1+c_3} = \gamma_1^2 + \gamma_2^2, \qquad \frac{-2a_3}{1+c_3} = 2\gamma_1, \qquad \frac{-2b_3}{1+c_3} = -2\gamma_2.$$

Hence

$$\gamma_1 = -\frac{a_3}{1+c_3}, \qquad \gamma_2 = \frac{b_3}{1+c_3}, \qquad \gamma_1^2 + \gamma_2^2 = \frac{1-c_3}{1+c_3}.$$

These last three equations are consistent; for, from the first two

$$\gamma_1^2 + \gamma_2^2 = \frac{a_3^2 + b_3^2}{(1+c_3)^2} = \frac{1-c_3^2}{(1+c_3)^2} = \frac{1-c_3}{1+c_3},$$

which gives the third equation.

Again, considering the two sets of numerators, we must have

$$\alpha_1\gamma_1 + \alpha_2\gamma_2 = \frac{c_1}{1+c_3}, \qquad \alpha_2\gamma_1 - \alpha_1\gamma_2 = \frac{c_2}{1+c_3},$$

from which we obtain

$$\alpha_1 = -\frac{a_3c_1 + b_3c_2}{1-c_3^2}, \qquad \alpha_2 = \frac{b_3c_1 - a_3c_2}{1-c_3^2}.$$

And, again,

$$\beta_1 = -\frac{c_1}{1+c_3}, \qquad \beta_2 = -\frac{c_2}{1+c_3}.$$

Finally, these values of the α's, β's, and γ's must also satisfy the four remaining equations,

$$\alpha_1 + \beta_1\gamma_1 + \beta_2\gamma_2 = \frac{2a_1}{1+c_3}, \qquad -\alpha_2 - \beta_1\gamma_2 + \beta_2\gamma_1 = \frac{2b_1}{1+c_3},$$

$$\alpha_2 + \beta_2\gamma_1 - \beta_1\gamma_2 = \frac{2b_2}{1+c_3}, \qquad \alpha_1 - \beta_1\gamma_1 - \beta_2\gamma_2 = \frac{2b^2}{1+c_3}.$$

Now

$$\beta_1\gamma_1 + \beta_2\gamma_2 = \frac{a_3c_1 - b_3c_2}{(1+c_3)^2},$$

and

$$\beta_1\gamma_2 - \beta_2\gamma_1 = -\frac{b_3c_1 + c_2a_3}{(1+c_3)^2}.$$

Consequently, we must have

$$-\frac{a_3c_1 + b_3c_2}{1-c_3^2} + \frac{a_3c_1 + b_3c_2}{(1+c_3)^2} = \frac{2a_1}{1+c_3},$$

$$-\frac{a_3c_2 - b_3c_1}{1-c_3^2} + \frac{b_3c_1 + c_2a_3}{(1+c_3)^2} = \frac{2a_2}{1+c_3},$$

$$\frac{a_3c_2 - b_3c_1}{1-c_3^2} + \frac{b_3c_1 + c_2a_3}{(1+c_3)^2} = \frac{2b_1}{1+c_3},$$

$$-\frac{a_3c_1 + b_3c_2}{1-c_3^2} - \frac{a_3c_1 - b_3c_2}{(1+c_3)^2} = \frac{2b_2}{1+c_3};$$

or

$$-b_3c_2 - a_3c_1c_3 = a_1(1-c_3^2), \qquad a_3c_2 - b_3c_1c_3 = b_1(1-c_3^2),$$

$$b_3c_1 - a_3c_2c_3 = a_2(1-c_3^2), \qquad -a_3c_1 - b_3c_2c_3 = b_2(1-c_3^2);$$

or again

$$c_3(a_1c_3 - a_3c_1) = a_1 + b_3c_2, \qquad c_3(b_1c_3 - b_3c_1) = b_1 - a_3c_2,$$

$$c_3(a_2c_3 - a_3c_2) = a_2 - b_3c_1, \qquad c_3(b_2c_3 - b_3c_2) = b_2 + a_3c_1;$$

or

$$c_3b_2 - b_3c_2 = a_1, \qquad -c_3a_2 + a_3c_2 = b_1,$$

$$-c_3b_1 + b_3c_1 = a_2, \qquad c_3a_1 - c_1a_3 = b_2;$$

which are known identities.

The collineation

$$\xi' = a_1\xi + b_1\eta + c_1\zeta,$$

$$\eta' = a_2\xi + b_2\eta + c_2\zeta,$$

$$\zeta' = a_3\xi + b_3\eta + c_3\zeta;$$

where

$$a_1^2 + a_2^2 + a_3^2 = 1, \qquad a_1b_1 + a_2b_2 + a_3b_3 = 0,$$

$$b_1^2 + b_2^2 + b_3^2 = 1, \qquad b_1c_1 + b_2c_2 + b_3c_3 = 0,$$

$$c_1^2 + c_2^2 + c_3^2 = 1; \qquad a_1c_1 + a_2c_2 + a_3c_3 = 0;$$

and

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = +1,$$

is equivalent in its effect on the spherical surface to the linear transformation of the complex variable

$$z' = \frac{\dfrac{[-(a_3c_1 + b_3c_2) + (b_3c_1 - a_3c_2)\,i\,]\,z}{1 - c_3^2} - \dfrac{c_1 + c_2 i}{1 + c_3}}{-\dfrac{a_3 - b_3 i}{1 + c_3}\,z + 1},$$

or

$$z' = \frac{[-(a_3c_1 + b_3c_2) + (b_3c_1 - a_3c_2)\,i\,]\,z - (c_1 + c_2 i)(1 - c_3)}{(a_3 - b_3 i)(c_3 - 1)\,z + 1 - c_3^2}.$$

The reader may show in the same way that if the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -1,$$

the corresponding collineation is reducible to the conjugate transformation,

$$\bar{z}' = \frac{[-(a_3c_1 - b_3c_2) + (b_3c_1 + a_3c_2)\,i\,]\,z - (c_1 - c_2 i)(1 - c_3)}{(a_3 - b_3 i)(c_3 - 1)\,z + 1 - c_3^2}.$$

80. In closing this article, mention must be made of another geometrical apparatus devised by Carl Neuman to illustrate the theory of linear transformation. This is a second plane tangent to the complex sphere at its north pole. If, now, we suppose the complex numbers to be distributed as usual on the spherical surface, and if we project this distribution stereographically from the south pole of the sphere on the new tangent plane, we obtain a new distribution of the complex numbers in this plane. We assert that this new distribution is, if the axes in the plane be properly chosen, the reciprocal of the distribution in the original plane.

The demonstration follows immediately from the fact, that the operation of reciprocation consists in a rotation of the sphere about the axis of ξ. This rotation interchanges the north and south poles of the sphere, and with these their tangent planes. That which was the projection from the

north pole on the tangent plane at the south pole becomes the projection from the transformed south pole on the tangent plane at the north pole.

It must, however, be noted that while this rotation leaves the direction of the axis of x unchanged, that of y is reversed. Consequently, if we look at the new tangent plane from above, the axis of y is drawn so as to make an angle of $-90°$ with the axis of x. As seen from the centre of the sphere, however, both axes have their normal positions.

NOTES.

The double spiral was introduced by Holzmüller, Zeitschrift für Math. und Physik, Bd. 16. See also Theorie der Isogonalen Verwandtschaften by the same author.

The orthogonal substitutions have been treated by Cayley, Crelle, Bd. 32. Cf. Salmon's Higher Algebra, p. 41, and Scott's Determinants. Cayley gave a method of expressing the 9 constants of such a transformation rationally in terms of the three essential constants.

The theory of rotations was also treated by Rodrigues, Liouville's Journal, Vol. 5, and in connection with the linear transformations of a complex variable by Cayley, Math. Ann., Bd. XV. See also Klein, Ikosaeder, Chap. II.

The arrangement of much of the present article follows closely that of Klein's Lectures on the Theory of Functions, Leipzig, 1880–'81.

EXERCISE.

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INTEGRATE $y\left(1+\dfrac{d^2y}{dx^2}\right)=x.$ [*R. S. Woodward.*]